RECEIVED: Sep 08, 2019
ACCEPTED: Dec 25, 2019
PUBLISHED: Jan 18, 2020

Online Mathematics
DOI: 10.5281/zenodo.3633891
ISSN: 2672-7501
http://www. onlinemathematics.net

RESEARCH ARTICLE

STABILITY, EXISTENCE AND UNIQUENESS OF A COUPLED SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

Areen Al-Khateeb[1,] Ayman A.hazaymeh [2], Mowafaq al-tahat [3]

[1] Jadara University, Irbid, Jordan; areen.k@jadara.edu.jo;
[2] Jadara University, Irbid, Jordan; aymanha@jadara.edu.jo;
[3] Jadara University, Irbid, Jordan; mwafag.tahat1997@gmail.com.

* Correspondence: areen.k@jadara.edu.jo

Abstract

In this article a coupled system of fractional differential equations with integral boundary conditions will be discussed. Currently we present three main result of this study: firstly, the uniqueness of solution for the given problem is established by applying contraction mapping principle. Secondly,Leray-Schauder's alternative has been used to obtain the existence of solutions. Moreover, some necessary conditions for theHyers-Ulamtype stability to the solutions of the boundary value problem (BVPs) are developed. Finally the results are supported by examples.

Keywords: Fractional Derivative, Fixed Point Theorem, Fractional Differential Equation

1. Introduction

Fractional differential equations and their boundary value problems are amongst the topics of modern mathematics that have markedly grown in popularity due to their numerous applications in various fields of engineering and natural sciences. One of the broad applications of fractional differential equations is in mathematical modeling of complicated systems, and the field of impulsive fractional differential equations has recently stood out as a promising field in this regard. Over the past few years, impulsive fractional differential equations have been comprehensively studied and generated a notable body of literature with broad potential future applications. The following section briefly reviews some of the recent studies on this topic.
In another study on coupled systems of fractional differential equations with nonlocal integral boundaries, Ntouyas and Obaid [3] used the Schauder alternative and Banach's fixed-point theorem to prove the existence and uniqueness of solutions for the coupled fractional differential equations with Riemann-Liouville integral boundary conditions, given below:

RECEIVED: Sep 08, 2019
ACCEPTED: Dec 25, 2019
PUBLISHED: Jan 18, 2020

Online Mathematics
DOI: 10.5281/zenodo.3633891
ISSN: 2672-7501
http://www. onlinemathematics.net

We study a coupled system of nonlinear fractional differential equations in this paper:

$$\begin{cases} {}^cD^\alpha u(t) = f\big(t, u(t), v(t)\big), & t \in [0,T], \quad 1 < \alpha \le 2, \\ {}^cD^\beta v(t) = g\big(t, u(t), v(t)\big), & t \in [0,T], \quad 1 < \beta \le 2, \end{cases} \tag{1}$$

Supplemented with boundary conditions of the form:

$$\begin{cases} u(0) = \lambda v'(T), & v(0) = \mu u'(T) \\ u'(0) = 0, & v'(0) = 0 \end{cases} \tag{2}$$

where ${}^cD^k$ denote the Caputo fractional derivatives of order $k, k = \alpha, \beta$, and $f, g \in C([0,T] \times \mathbb{R}^2, \mathbb{R})$ are given continuous functions, and λ, μ are real constants.

2. Preliminaries

Firstly, we recall definitions of fractional derivative and integral [1, 2].

Definition 2.1.*The Riemann-Liouville fractional integral of order q for a continuous function g is given by*

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \qquad q > 0,$$

Provided that the right-hand side is point-wise defined on $[0, \infty)$.

Definition 2.2.*The Caputo fractional derivatives of order q for $(g-1)$ −times absolutely continuous function $h: [0,\infty) \to \mathbb{R}$ is defined as*

$${}^cD^q h(t) = \frac{1}{\Gamma(g-q)} \int_0^t (t-s)^{g-q-1} h^{(g)}(s) ds, \quad g-1 < q < g, \quad g = [q]+1,$$

where$[q]$ is the integer part of real number q.

Lemma 2.3 *Let $x, y \in C([0,T], \mathbb{R})$ then the unique solution for the problem*

RECEIVED: Sep 08, 2019
ACCEPTED: Dec 25, 2019
PUBLISHED: Jan 18, 2020

DOI: 10.5281/zenodo.3633891
ISSN: 2672-7501
http://www. onlinemathematics.net

$$\begin{cases} {}^cD^{\alpha}u(t) = x(t), \quad t \in [0,T], \;\; 1 < \alpha \leq 2, \\ {}^cD^{\beta}v(t) = y(t), \quad t \in [0,T], \;\; 1 < \beta \leq 2, \\ u(0) = \lambda v'(T), \qquad v(0) = \mu u'(T) \\ u'(0) = 0, \qquad v'(0) = 0 \end{cases} \tag{3}$$

is

$$u(t) = \lambda \frac{1}{\Gamma(\beta-1)} \int_0^T (T-s)^{\beta-2}\, y(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, x(s)ds \tag{4}$$

and

$$v(t) = \mu \frac{1}{\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2}\, x(s)ds + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\, y(s)ds \tag{5}$$

Proof. The general solutions of the fractional differential equations in (3) are known [6] as

$$u(t) = at + b + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, x(s)ds, \tag{6}$$

$$v(t) = ct + d + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\, y(s)ds, \tag{7}$$

where a, b, c, d are arbitrary constants.

Here

$$u'(t) = a + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2}\, x(s)ds,$$

$$v'(t) = c + \frac{1}{\Gamma(\beta-1)} \int_0^t (t-s)^{\beta-2}\, y(s)ds.$$

Apply the conditions $u'(0) = 0,\; v'(0) = 0$ then we obtain $a = c = 0$.

In view of the conditions

RECEIVED: *Sep 08, 2019*
ACCEPTED: *Dec 25, 2019*
PUBLISHED: *Jan 18, 2020*

Online Mathematics
DOI: 10.5281/zenodo.3633891
ISSN: 2672-7501
http://www. onlinemathematics.net

$$u(0) = \lambda v'(T), \qquad v(0) = \mu u'(T)$$

we get

$$b = \lambda \frac{1}{\Gamma(\beta - 1)} \int_0^T (T - s)^{\beta-2} \, y(s) ds,$$

And

$$d = \mu \frac{1}{\Gamma(\alpha - 1)} \int_0^T (T - s)^{\alpha-2} \, x(s) ds,$$

,

Substituting the values of a, b, c, d in (6), (7) we get (4) and (5).The converse follows by direct computation. This completes the proof.■

3. Existence results

Let us consider the space

$$W = \{u(t), \qquad u(t) \in C([0, T]\},$$

O

$$Z = \{v(t), \qquad v(t) \in C([0, T]\},$$

endowed with norm $\|u\| = \sup_{0 \le t \le T} |u(t)|$ and $\|v\| = \sup_{0 \le t \le T} |v(t)|$ respectively.

It is clear that both $(W, \| \,.\, \|)$ and $(Z, \| \,.\, \|)$ are Banach Spaces.

Consequently, the product space$(W \times Z, \|(u, v)\|)$ is a Banach Space as well (endowed with $\|(u, v)\| = \|u\| + \|v\|$).

In view of Lemma (2..3), we define the operator $Q: W \times Z \to W \times Z$ as:

$$Q(u, v)(t) = \big(Q_1(u, v)(t), Q_2(u, v)(t)\big),$$

where

RECEIVED: Sep 08, 2019
ACCEPTED: Dec 25, 2019
PUBLISHED: Jan 18, 2020

Online Mathematics
DOI: 10.5281/zenodo.3633891
ISSN: 2672-7501
http://www. onlinemathematics.net

$$Q_1(u,v)(t) = \lambda \frac{1}{\Gamma(\beta-1)} \int_0^T (T-s)^{\beta-2}\, g\big(s, x(s), y(s)\big) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, f\big(s, x(s), y(s)\big) ds$$

and

$$Q_2(u,v)(t) = \mu \frac{1}{\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2}\, f\big(s, x(s), y(s)\big) ds + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\, g\big(s, x(s), y(s)\big) ds.$$

In the first result, we establishes the existence and the uniqueness of the solutions of the boundary value problem (1) and (2) by using Banach's contraction mapping principle.

Theorem 3.1: *Assume* $f, g\colon C([0,T] \times \mathbb{R}^2 \to \mathbb{R}$ *are jointly continuous functions and there exist constants* $\rho, \eta \in \mathbb{R}$ *,such that* $\forall\, u_1, u_2, v_1, v_2 \in \mathbb{R}, \forall t \in [0,T]$, *we have*

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \le \rho(|u_2-u_1| + |v_2-v_1|),$$

$$|g(t,u_1,u_2) - f(t,v_1,v_2)| \le \eta(|u_2-u_1| + |v_2-v_1|).$$

If

$$\rho(G_1+G_3) + \eta(G_2+G_4) < 1,$$

then the BVP (1) and (2) has a unique solution on $[0,T]$.

Where

$$G_1 = \frac{T^\alpha}{\Gamma(\alpha+1)},$$

$$G_2 = \frac{|\lambda| T^{\beta-1}}{\Gamma(\beta)},$$

$$G_3 = \frac{|\mu| T^{\alpha-1}}{\Gamma(\alpha)},$$

$$G_4 = \frac{T^\beta}{\Gamma(\beta+1)}.$$

RECEIVED: *Sep 08, 2019*
ACCEPTED: *Dec 25, 2019*
PUBLISHED: *Jan 18, 2020*

Online Mathematics
DOI: 10.5281/zenodo.3633891
ISSN: 2672-7501
http://www. onlinemathematics.net

Proof: Define $\sup_{0\le t\le T} |f(t,0,0)| = f_0 < \infty$, $\sup_{0\le t\le T} |g(t,0,0)| = g_0 < \infty$and $\Omega_r = \{(u,v) \in W \times Z : \|(u,v)\| \le r\}$, and $r > 0$, such that

$$r \ge \frac{(G_1 + G_3)f_0 + (G_2 + G_4)g_0}{1 - [\rho(G_1 + G_3) + \eta(G_2 + G_4)]}.$$

Firstly, we show that $Q\Omega_r \subseteq \Omega_r$.

By our assumption, for $(u,v) \in \Omega_r, t \in [0,T]$, we have

$$\left|f\left(t,u(t),v(t)\right)\right| \le \left|f\left(t,u(t),v(t)\right) - f(t,0,0)\right| + |f(t,0,0)|,$$

$$\le \rho(|u(t)| + |v(t)|) + f_0 \le \rho(\|u\| + \|v\|) + f_0,$$

$$\le \rho r + f_0,$$

and

$$\left|g\left(t,u(t),v(t)\right)\right| \le \eta(|u(t)| + |v(t)|) + g_0 \le \eta(\|u\| + \|v\|) + g_0,$$

$$\le \eta r + g_0,$$

which lead to

$$|Q_1(u,v)(t)| \le |\lambda| \frac{1}{\Gamma(\beta-1)} \int_0^T (T-s)^{\beta-2} \left(\eta(\|u\| + \|v\|) + g_0\right) ds + \sup_{0 \le t \le T} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\rho(\|u\| + \|v\|) + f_0\right) ds$$

$$\le (\rho(\|u\| + \|v\|) + f_0)\left[\frac{T^\alpha}{\Gamma(\alpha+1)}\right] + (\eta(\|u\| + \|v\|) + g_0)\left[\frac{|\lambda| T^{\beta-1}}{\Gamma(\beta)}\right],$$

$$\le (\rho(\|u\| + \|v\|) + f_0)G_1 + (\eta(\|u\| + \|v\|) + g_0)G_2,$$

$$\le (\rho r + f_0)G_1 + (\eta r + g_0)G_2.$$

In a like manner

$$|Q_2(u,v)(t)| \le (\rho(\|u\| + \|v\|) + f_0)G_3 + (\eta(\|u\| + \|v\|) + g_0)G_4$$

RECEIVED: Sep 08, 2019
ACCEPTED: Dec 25, 2019
PUBLISHED: Jan 18, 2020

Online Mathematics
DOI: 10.5281/zenodo.3633891
ISSN: 2672-7501
http://www. onlinemathematics.net

$$\leq (\rho r + f_0)G_3 + (\eta r + g_0)G_4.$$

Hence

$$\|Q_1(u,v)\| \leq (\rho r + f_0)G_1 + (\eta r + g_0)G_2,$$

and

$$\|Q_2(u,v)\| \leq (\rho r + f_0)G_3 + (\eta r + g_0)G_4.$$

Consequently,

$$\|Q(u,v)\| \leq (\rho r + f_0)(G_1 + G_3) + (\eta r + g_0)(G_2 + G_4) \leq r\ .$$

So we get $\|Q(u,v)\| \leq r$ that is $Q\Omega_r \subseteq \Omega_r$.

Now let $(u_1, v_1), (u_2, v_2) \in W \times Z, \forall t \in [0,T]$ then we get

$$\begin{aligned}|Q_1(u_1,v_1)(t) - Q_1(u_2,v_2)(t)| &\leq |\lambda| \frac{1}{\Gamma(\beta-1)} \int_0^T (T-s)^{\beta-2} |g(s,x_1,x_2) - g(s,y_1,y_2)| ds \\ &+ \sup_{0 \leq t \leq T} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s,x_1,x_2) \\ &- f(s,y_1,y_2)| ds\end{aligned}$$

So

$$\begin{aligned}|Q_1(u_1,v_1)(t) - Q_1(u_2,v_2)(t)| &\leq |\lambda| \frac{1}{\Gamma(\beta-1)} \int_0^T (T-s)^{\beta-2} \eta(\|u_2 - u_1\| + \|v_2 - v_1\|) ds \\ &+ \sup_{0 \leq t \leq T} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \rho(\|u_2 - u_1\| \\ &+ \|v_2 - v_1\|) ds\end{aligned}$$

$$\begin{aligned}\|Q_1(u_1,v_1)(t) - Q_1(u_2,v_2)(t)\| \\ \leq G_1\rho(\|u_2 - u_1\| + \|v_2 - v_1\|) + G_2\eta(\|u_2 - u_1\| + \|v_2 - v_1\|). \qquad (8)\end{aligned}$$

Similarly

RECEIVED: *Sep 08, 2019*
ACCEPTED: *Dec 25, 2019*
PUBLISHED: *Jan 18, 2020*

Online Mathematics
DOI: 10.5281/zenodo.3633891
ISSN: 2672-7501
http://www. onlinemathematics.net

$$\|Q_2(u_1,v_1)(t)-Q_2(u_2,v_2)(t)\| \leq G_3\rho(\|u_2-u_1\|+\|v_2-v_1\|)+G_4\eta(\|u_2-u_1\|+\|v_2-v_1\|). \quad (9)$$

From (8) and (9) we deduced that

$$\|Q(u_1,v_1)-Q(u_2,v_2)\| \leq \big(\rho(G_1+G_3)+\eta(G_2+G_4)\big)(\|u_2-u_1\|+\|v_2-v_1\|).$$

Since $\rho(G_1+G_3)+\eta(G_2+G_4)<1$, therefore, the operator Q is a contraction operator. Hence, by Banach's fixed-point theorem, the operator Q is has unique fixed point on, which is the unique solution of BVP (1) and (2) .This completes the proof.■

The next result is based on the Leray-Schauder alternative.

Lemma 3.2. *"(Leray-Schauder alternative [7], p.4) Let $F: E \to E$ be a completely continuous operator (i.e., a map restricted to any bounded set in E is compact).Let $E(F) = \{x \in E: x = \lambda F(x)\ for\ some\ 0<\lambda<1\}$. Then either the set $E(F)$ is unbounded or F has at least one fixed point."*

Theorem 3.3. *Assume $f, g: C([0,T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous function and there exist*

$\phi_1,\phi_2,\psi_1,\psi_2 \geq 0$ where $\phi_1,\phi_2,\psi_1,\psi_2$ are real constants and $\phi_0,\psi_0>0$ such that $\forall u_i, v_i \in \mathbb{R}, (i=1,2)$, we have

$$|f(t,u_1,u_2)| \leq \phi_0+\phi_1|u_1|+\phi_2|u_2|,$$

$$|g(t,u_1,u_2)| \leq \psi_0+\psi_1|u_1|+\psi_2|u_2|,$$

If

$$(G_1+G_3)\phi_1+(G_2+G_4)\psi_1<1,$$

and

$$(G_1+G_3)\phi_2+(G_2+G_4)\psi_2<1,$$

where $G_i, i=1,2,3,4$ are defined before, then the problem (1) and (2) has at least one solution.

Proof. The proof will be divided into several steps

Step1:show that $Q: W \times Z \to W \times Z$ is completely continuous .The continuity of the operator Q holds by the continuity of the functions f, g .

Let $R \subseteq W \times Z$ be a bounded. Then there exists positive constants k_1, k_2 such that

RECEIVED: Sep 08, 2019
ACCEPTED: Dec 25, 2019
PUBLISHED: Jan 18, 2020

Online Mathematics
DOI: 10.5281/zenodo.3633891
ISSN: 2672-7501
http://www. onlinemathematics.net

$$\left|f\big(t,u(t),v(t)\big)\right| \le k_1, \qquad \left|g\big(t,u(t),v(t)\big)\right| \le k_2, \quad \forall t \in [0,T].$$

Then $\forall(u,v) \in R$, we have

$$|Q_1(u,v)(t)| \le G_1k_1 + G_2k_2,$$

Which implies that

$$\|Q_1(u,v)\| \le G_1k_1 + G_2k_2,$$

Similarly, we get

$$\|Q_2(u,v)\| \le G_3k_1 + G_4k_2,$$

Thus, from the above inequalities, it follows that the operator Q is uniformly bounded, since

$$\|Q(u,v)\| \le (G_1 + G_3)k_1 + (G_2 + G_4)k_2.$$

Further, we show that the operator Q is equicontinuous.Let $\mathrm{t}_1, \mathrm{t}_2 \in [0,T]$ with $\mathrm{t}_1 < \mathrm{t}_2$.This yields

$$\begin{aligned}
&|Q_1(u,v)(\mathrm{t}_2) - Q_1(u,v)(\mathrm{t}_1)| \\
&\qquad \le \int_0^{\mathrm{t}_2} \frac{|\mathrm{t}_2 - s|^{\alpha-1}}{\Gamma(\alpha)} \left|f\big(s,x(s),y(s)\big)\right| ds \\
&\qquad + \int_0^{\mathrm{t}_1} \frac{|\mathrm{t}_1 - s|^{\alpha-1}}{\Gamma(\alpha)} \left|f\big(s,x(s),y(s)\big)\right| ds,
\end{aligned}$$

$$\le k_1\left(\int_0^{t_2} \frac{|t_2 - s|^{\alpha-1}}{\Gamma(\alpha)} ds + \int_0^{t_1} \frac{|t_1 - s|^{\alpha-1}}{\Gamma(\alpha)} ds\right),$$

$$\le \frac{k_1}{\Gamma(\alpha)}\left(\int_0^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}|\, ds + \int_{t_1}^{t_2} |t_2 - s|^{\alpha-1}\, ds\right).$$

We can obtain

$$|Q_1(u,v)(t_2) - Q_1(u,v)(t_1)| \le \frac{k_1}{\Gamma(\alpha+1)}[|(t_2 - t_1)^{\alpha} - {t_2}^{\alpha}| + {t_1}^{\alpha} + |t_2 - t_1|^{\alpha}].$$

Hence we have $\|Q_1(u,v)(t_2) - Q_1(u,v)(t_1)\| \to 0$ independent of u and v as $t_2 \to t_1$.Also, we can obtain

$$|Q_2(u,v)(t_2) - Q_2(u,v)(t_1)| \le \frac{k_2}{\Gamma(\beta+1)}\left[\left|(t_2 - t_1)^{\beta} - {t_2}^{\beta}\right| + {t_1}^{\beta} + |t_2 - t_1|^{\beta}\right],$$

RECEIVED: *Sep 08, 2019*
ACCEPTED: *Dec 25, 2019*
PUBLISHED: *Jan 18, 2020*

Online Mathematics
DOI: 10.5281/zenodo.3633891
ISSN: 2672-7501
http://www. onlinemathematics.net

which implies that $\|Q_2(u,v)(t_2) - Q_2(u,v)(t_1)\| \to 0$ independent of u and v as $t_2 \to t_1$.

Therefore, the operator $Q(u,v)$ is equicontinuous, and thus the operator $Q(u,v)$ is completely continuous.

Step 2: (Boundedness of operator)

Finally, show that $S = \{(u,v) \in W \times Z: (u,v) = \delta Q(u,v), \delta \in [0,1]\}$ is bounded. Let $(u,v) \in \mathbb{R}$, with $(u,v) = \delta Q(u,v)$ for any $t \in [0,T]$, we have

$$u(t) = \delta Q_1(u,v)(t), \qquad v(t) = \delta Q_2(u,v)(t).$$

Then

$$|u(t)| \le G_1(\phi_0 + \phi_1|u| + \phi_2|v|) + G_2(\psi_0 + \psi_1|u| + \psi_2|v|),$$

and

$$|v(t)| \le G_3(\phi_0 + \phi_1|u| + \phi_2|v|) + G_4(\psi_0 + \psi_1|u| + \psi_2|v|).$$

So we get

$$\|u\| \le G_1(\phi_0 + \phi_1\|u\| + \phi_2\|v\|) + G_2(\psi_0 + \psi_1\|u\| + \psi_2\|v\|),$$

and

$$\|y\| \le G_3(\phi_0 + \phi_1\|u\| + \phi_2\|v\|) + G_4(\psi_0 + \psi_1\|u\| + \psi_2\|v\|),$$

which imply that

$$\begin{aligned}\|u\| + \|v\| \le (G_1 + G_3)\phi_0 + (G_2 + G_4)\psi_0 + \big((G_1 + G_3)\phi_1 + (G_2 + G_4)\psi_1\big)\|u\| \\ + \big((G_1 + G_3)\phi_2 + (G_2 + G_4)\psi_2\big)\|v\|.\end{aligned}$$

Therefore,

$$\|(u,v)\| \le \frac{(G_1 + G_3)\phi_0 + (G_2 + G_4)\psi_0}{G_0},$$

where$G_0 = min\{1 - (G_1 + G_3)\theta_1 - (G_2 + G_4)\psi_1, 1 - (G_1 + G_3)\phi_2 - (G_2 + G_4)\psi_2\}$, which proves that S is bounded. By (Leray-Schauder theorem) the operator Q has at least one fixed point. Therefore, the BVP (1) and (2) has at least one solution on $[0,T]$.The proof is complete.■

RECEIVED: *Sep 08, 2019*
ACCEPTED: *Dec 25, 2019*
PUBLISHED: *Jan 18, 2020*

Online Mathematics
DOI: 10.5281/zenodo.3633891
ISSN: 2672-7501
http://www. onlinemathematics.net

4. Examples

Example 4.1.consider the following system of fractional differential equation

$$\begin{cases} {}^cD^{5/4}u(t) = \dfrac{1}{4\pi\sqrt{64+t^2}}\left(\dfrac{|u(t)|}{5+|u(t)|}+\dfrac{|v(t)|}{3+|u(t)|}\right), \\ {}^cD^{4/3}v(t) = \dfrac{1}{2\pi\sqrt{81+t^2}}\left(sin\big(u(t)\big)+sin\big(v(t)\big)\right), \\ u(0)=-v'(1), \qquad v(0)=-u'(1) \\ u'(0)=0, \qquad v'(0)=0, \end{cases} \tag{10}$$

$$\alpha=\frac{5}{4},\beta=\frac{4}{3},T=1,\lambda=-1,\mu=-1.$$

Using the given data , we find that $G_1 = 0.8826, G_2 = 1.119, G_3 = 1.1032, G_4 = 0.8399, \rho = \frac{1}{32\pi}, \eta = \frac{1}{18\pi}$.

It's clear that

$$f\big(t,u(t),v(t)\big)=\frac{1}{4\pi\sqrt{64+t^2}}\left(\frac{|u(t)|}{5+|u(t)|}+\frac{|v(t)|}{3+|u(t)|}\right),$$

and

$$g\big(t,u(t),v(t)\big)=\frac{1}{2\pi\sqrt{81+t^2}}\left(sin\big(u(t)\big)+sin\big(v(t)\big)\right),$$

are jointly continuous functions and $\rho(G_1+G_3)+\eta(G_2+G_4)<1$, such that

$$\frac{1}{32\pi}(0.8826+1.1032)+\frac{1}{18\pi}(1.119+0.8399)=0.0544<1.$$

Thus all the conditions of Theorem 3.1 are satisfied, then problem (10) has a unique solution on [0,1].

Example 4.2.consider the following system of fractional differential equation

RECEIVED: *Sep 08, 2019*
ACCEPTED: *Dec 25, 2019*
PUBLISHED: *Jan 18, 2020*

Online Mathematics
DOI: 10.5281/zenodo.3633891
ISSN: 2672-7501
http://www. onlinemathematics.net

$$\begin{cases} {}^{c}D^{5/4}u(t) = \dfrac{1}{60+t^2} + \dfrac{|u(t)|}{100\left(1+v^2(t)\right)} + \dfrac{1}{9\sqrt{6400+t^4}} e^{-3t}\cos\left(v(t)\right), t \in [0,1] \\ {}^{c}D^{4/3}v(t) = \dfrac{1}{\sqrt{25+t^4}}\cos t + \dfrac{1}{130} e^{-3t} sin\left(v(t)\right) + \dfrac{1}{160}u(t), t \in [0,1] \\ u(0) = -v'(1), \quad v(0) = -u'(1) \\ u'(0) = 0, \quad v'(0) = 0 \end{cases} \tag{11}$$

$$\alpha = \frac{5}{4}, \beta = \frac{4}{3}, T = 1, \lambda = -1, \mu = -1.$$

Using the given data, we find that $G_1 = 0.8826, G_2 = 1.119, G_3 = 1.1032, G_4 = 0.8399$

It is clear that

$$|f(t, u_1, u_2)| \leq \frac{1}{60} + \frac{1}{100}\|u\| + \frac{1}{720}\|v\|,$$

$$|g(t, u_1, u_2)| \leq \frac{1}{5} + \frac{1}{130}\|u\| + \frac{1}{160}\|v\|.$$

Thus$\phi_0 = \frac{1}{60}, \phi_1 = \frac{1}{100}, \phi_2 = \frac{1}{720}, \psi_0 = \frac{1}{5}, \psi_1 = \frac{1}{130}, \psi_2 = \frac{1}{160}$.

We found $(G_1 + G_3)\phi_1 + (G_2 + G_4)\psi_1 = 0.04815 < 1$ and $(G_1 + G_3)\phi_2 + (G_2 + G_4)\psi_2 = 0.01226 < 1$, then by Theorem 3.3 the problem (11) has at least one solution on $[0,1]$.

Author Contributions: The authors have made the same contribution. All authors read and approved the final manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflicts of interest.

Acknowledgments: The authors wish to thank the anonymous reviewers for their valuable commentsand suggestions.

Competing interests: The authors declare that they have no competing interests.

References

[1] Podlubny, I: Fractional Differential Equations. Academic Press, San Diego (1999)

[2] Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)

[3] S. K. Ntouyas and M. Obaid, "A coupled system of fractional differential equations with nonlocal integral boundary

RECEIVED: *Sep 08, 2019*
ACCEPTED: *Dec 25, 2019*
PUBLISHED: *Jan 18, 2020*

Online Mathematics
DOI: 10.5281/zenodo.3633891
ISSN: 2672-7501
http://www. onlinemathematics.net

conditions," Advances in Difference Equations, vol. 2012, no. 1, pp. 130-139, 2012.

[4] B. Ahmad and S. K. Ntouyas, "A Coupled System of Nonlocal Fractional Differential Equations with Coupled and Uncoupled Slit-Strips-Type Integral Boundary Conditions," Journal of Mathematical Sciences, vol. 226, no. 3, pp. 175-196, 2017.

[5] H. H. Alsulami, S. K. Ntouyas, R. P. Agarwal, B. Ahmad and A. Alsaedi, "A study of fractional-order coupled systems with a new concept of coupled non-separated boundary conditions," Boundary Value Problems, vol. 2017, no. 1, pp. 68-74, 2017.

[6] Zhang, Y., Bai, Z., & Feng, T. Existence results for a coupled system of nonlinear fractional three-point boundary value problems at resonance. Computers & Mathematics with Applications, 2011. , 61(4), 1032-1047.

[7] Granas, A, Dugundji, J: Fixed Point Theory. Springer, New York (2005)/

[8] Hyers, Donald H. "On the stability of the linear functional equation." Proceedings of the National Academy of Sciences of the United States of America 27.4 (1941): 222.

[9] Rus, Ioan A. "Ulam stabilities of ordinary differential equations in a Banach space." Carpathian journal of Mathematics(2010): 103-107.

[10] Cabada, Alberto, and Guotao Wang. "Positive solutions of nonlinear fractional differential equations with integral boundary value conditions." Journal of Mathematical Analysis and Applications 389.1 (2012): 403-411.

[11] Graef, John R., Lingju Kong, and Min Wang. "Existence and uniqueness of solutions for a fractional boundary value problem on a graph." Fractional Calculus and Applied Analysis17.2 (2014): 499-510.

[12] Ahmad, Bashir, and Juan J. Nieto. "Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions." Computers & Mathematics with Applications 58.9 (2009): 1838-1843.

[13] Su, Xinwei. "Boundary value problem for a coupled system of nonlinear fractional differential equations." Applied Mathematics Letters 22.1 (2009): 64-69.

[14] Wang, Jinhua, Hongjun Xiang, and Zhigang Liu. "Positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations." International Journal of Differential Equations (2010).

[15] Ntouyas, Sotiris K., and Mustafa Obaid. "A coupled system of fractional differential equations with nonlocal integral boundary conditions." Advances in Difference Equations 2012.1 (2012): 130.

[16] Ahmad, Bashir, Sotiris K. Ntouyas, and Ahmed Alsaedi. "On a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions." Chaos, Solitons & Fractals 83 (2016): 234-241.

[17] Zhai, Chengbo, and Li Xu. "Properties of positive solutions to a class of four-point boundary value problem of Caputo fractional differential equations with a parameter." Communications in Nonlinear Science and Numerical Simulation 19.8 (2014): 2820-2827.

[18] Ahmad, Bashir, and Sotiris K. Ntouyas. "Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions. Applied Mathematics and Computation 266 (2015): 615-622.

[19] Tariboon, Jessada, Sotiris K. Ntouyas, and WeerawatSudsutad. "Coupled systems of Riemann-Liouville fractional differential equations with Hadamard fractional integral boundary conditions." J. Nonlinear Sci. Appl 9.1 (2016): 295-308.