RECEIVED: *Sep 06, 2019*
ACCEPTED: *Sep 29, 2019*
PUBLISHED: *Sep 31, 2019*

Online Mathematics
R.N OMJV1I4Patton
ISSN: 2672-7501
http://www. onlinemathematics.net

The Mathematical Errors of Infinity

Anthony C. Patton,
San Diego, California, USA, pattonanthonyc@gmail.com

Key Words: infinity, potential infinity, cardinality, Cantor, Aristotle, one-to-one correspondence, continuum, numbers, Axiom of Infinity, Axiom of Power Set, infinitesimal, derivative, Hilbert's Hotel, Diagonal Argument.

Introduction

Philosophical analysis of infinity has identified multiple logical and metaphysical problems that warrant a response. However, mathematicians generally do not consider these criticisms valid or relevant and instead rely on supposed mathematical rigor to assess the validity or practical uses of infinity. Therefore, this paper will highlight three errors in the mathematics of infinity to address mathematicians on their own terms.

One-to-One Correspondence Methodology

Cantor argued that two infinite sets have the same size or cardinality if a one-to-one correspondence exists between them. Consider the following example of the even numbers and the natural numbers:

Even		Natural
2	=>	1
4	=>	2
6	=>	3
8	=>	4
10	=>	5
…		…

This display is frequently used to demonstrate that the two sets have the same cardinality. However, the mere fact that two sets can be paired up ad infinitum does not prove that they have the same cardinality or that cardinality is a meaningful term in the context of infinite sets, for two reasons. First, this display does not prove that we do not exhaust the even numbers before we exhaust the natural numbers, as we will see below. Second, it does not rule out the possibility that both sets are only potentially infinite, in the tradition of Aristotle, in which case Cantor's theory of infinite sets does not apply. A purveyor of potential infinity would agree that the pairing of the two sets continues ad infinitum, without additional commentary. The sun rotating around the earth or the earth rotating around the sun looks the same from our perspective on the surface of the earth but only one explanation is correct. Cantor's claim about pairing up infinite sets make sense only if we already presume the existence of infinite sets, which is circular

RECEIVED: *Sep 06, 2019*
ACCEPTED: *Sep 29, 2019*
PUBLISHED: *Sep 31, 2019*

Online Mathematics
R.N OMJV1I4Patton
ISSN: 2672-7501
http://www. onlinemathematics.net

Online Mathematics

reasoning that is merely assumed to be true with the Axiom of Infinity in set theory. Therefore, we should consider a different pairing methodology to give us a more rigorous proof or insights of how the two sets actually compare in size, such as mapping the even numbers to their identical elements in the natural numbers:

Even		Natural
		1
2	=>	2
		3
4	=>	4
		5
6	=>	6
		7
8	=>	8
		9
10	=>	10
…		…

In this case, both sets might be infinite and the pairing process might continue ad infinitum, but with mathematical rigor we have shown without any ambiguity that we can exhaust the even numbers without exhausting the natural numbers—thus, no one-to-one correspondence between the two sets. By definition, therefore, either the two sets do not have the same cardinality or cardinality does not apply to infinite sets. Although it makes sense to say the natural numbers remain an infinite set after we remove the even numbers, it does not make sense to say the even numbers remain an infinite set after we remove the even numbers.

The same idea applies to the rational numbers, in two ways:

Natural		Rational	Natural		Rational
1	=>	1	1	=>	1
2	=>	2	2	=>	1/2
3	=>	3	3	=>	1/3
4	=>	4	4	=>	1/4
5	=>	5	5	=>	1/5
…		…	…		…

In the left example, we pair up the natural numbers with their identical elements in the rational numbers, as we did in the previous example. As should be clear, we exhaust the natural numbers but do not exhaust the rational numbers because we never list the infinite sets of fractions between any two natural numbers. In the right example, we pair up the natural numbers with their inverses. In this case, we exhaust the natural numbers but again do not exhaust the rational numbers because we miss numbers like 2, 3, 4, 2/3, 3/4, 4/5, and so on. Thus, not only are the natural numbers smaller than the set of rational numbers, the natural numbers are smaller than

RECEIVED: *Sep 06, 2019*
ACCEPTED: *Sep 29, 2019*
PUBLISHED: *Sep 31, 2019*

Online Mathematics
R.N OMJV1I4Patton
ISSN: 2672-7501
http://www. onlinemathematics.net

the set of rational numbers [0, 1].

We can use this same methodology for all sets and proper subsets, to include the natural numbers and the real numbers, by pairing up the natural numbers with their identical elements in the real numbers. Thus, either infinite sets do not exist (back to potential infinity, in which case cardinality does not apply to infinite sets) or all countable or denumerable sets do not have the same cardinality.

Filling the Gaps of the Continuum

Regarding the claim that the rational numbers never "fill the gaps" of the continuum, we can redefine the rational numbers as fractions with integer numerators and integer denominators of the form 10^x (1/10, 24/100, 328/1,000, and so on), expressed as decimals (0.1, 0.24, 0.328, and so on), to make an apples-to-apples comparison with the real numbers expressed as decimals. To exhaust the rational numbers [0, 1], the previous methodology (p/q) was as follows, with the repeated numbers eliminated:

1/2
1/3, 2/3
1/4, ~~2/4~~, 3/4
1/5, 2/5, 3/5, 4/5
1/6, ~~2/6~~, ~~3/6~~, ~~4/6~~, 5/6
…

This methodology, we are told, leaves "gaps" on the continuum that the irrational numbers "fill." The new methodology is as follows:

0.1, 0.2, 0.3, 0.4 … 0.9
0.01, 0.02, 0.03, 0.04 … 0.99
0.001, 0.002, 0.003, 0.004 … 0.999
0.0001, 0.0002, 0.0003, 0.0004 … 0.9999
0.00001, 0.00002, 0.00003, 0.00004 … 0.99999
…

With this methodology, we progress one decimal place at a time (tenths, hundredths, thousandths, and so on). This methodology is preferable because there are no duplicate entries. If we exhaust this methodology (to infinity), we generate all possible decimals, to include all terminating decimals, such as 0.25; all infinite repeating decimals, such as 0.333…; and all infinite nonrepeating decimals, such as $3 - \pi$. In this case, we use a general, exhaustive process to generate the set of infinite decimals rather than specific algorithms to generate specific infinite decimals.

Infinite decimals for rational numbers (repeating) and irrational numbers (nonrepeating) both have the same cardinality (the set of numbers after the decimal), which means that if this infinite,

RECEIVED: *Sep 06, 2019*
ACCEPTED: *Sep 29, 2019*
PUBLISHED: *Sep 31, 2019*

Online Mathematics
R.N OMJV1I4Patton
ISSN: 2672-7501
http://www. onlinemathematics.net

iterative process does not generate 3 - π, it does not generate 0.333… (1/3). This infinite, iterative process does not distinguish between rational and irrational numbers. With mathematical rigor, we see that the set of infinite decimals resulting from this infinite, iterative process is nothing other than the real numbers, which means our new definition of the rational numbers "fills the gaps" of the continuum (more on this later). However, if infinite sets do not exist (if we never exhaust this infinite, iterative process), then the real numbers do not exist and we are left with rational numbers.

One explanation for this result is that the previous definition of rational numbers (p/q) was a limiting factor. Just as we could not solve some polynomial equations before the invention of negative numbers and √-1 (i), perhaps our previous definition of rational numbers (p/q) prevented us from considering new numbers like infinite fractions. Historically, by comparing p/q fractions (rational numbers) to infinite nonrepeating decimals (irrational numbers), we were unable to "fill the gap," both mentally and symbolically, but this new methodology clears the way.

The Rules of Transfinite Arithmetic

The claim that the cardinality of the real numbers equals the power set of the natural numbers generates a contradiction when taken in combination with the rules of transfinite arithmetic, such as $\infty_0 + \infty_0 = \infty_0$, where ∞_0 is the cardinality of the natural numbers. According to Cantor, each unique real number (base-10) can be expressed as a unique infinite binary decimal (base-2). Given that the numbers after the decimals are denumerable, we can establish a one-to-one correspondence with the natural numbers, such that 1 corresponds to the tenth position, 2 corresponds to the hundredth position, 3 corresponds to the thousandths position, and so on. By limiting ourselves to 1s and 0s, we can thus generate the set of infinite binary decimals.

Natural numbers:	0.11111111111111111111…
Odd numbers:	0.10101010101010101010…
Even numbers:	0.01010101010101010101…
{1, 4, 7}:	0.10010010000000000000…
{5}:	0.00001000000000000000…
…	…

If we consider all the possible subsets of the natural numbers (represented as the set of infinite binary decimals) we generate a set with cardinality equal to the power set of the natural numbers. For all sets, if a set has n elements, the power set has 2^n elements, which is larger than the original set. In the case of infinite sets, we will use the notation 2^∞ or ∞_1. Each decimal place in a binary decimal has 2 options (1 or 0), which is repeated n times—thus, 2^n, or 2^∞ for the set of infinite binary decimals. (The cardinality of base-10 decimals would be 10^∞. Thus, Cantor needs a new infinite hotel with 2^∞ rooms that can accommodate 10^∞ guests.) To summarize, Cantor claimed that each unique real number (base-10) can be expressed as a unique binary decimal (base-2). If true, then the cardinality of the real numbers equals the power set of the natural numbers (2^∞).

RECEIVED: *Sep 06, 2019*
ACCEPTED: *Sep 29, 2019*
PUBLISHED: *Sep 31, 2019*

Online Mathematics
R.N OMJV1I4Patton
ISSN: 2672-7501
http://www. onlinemathematics.net

The proof of this is beyond the scope of this paper but the result generates a contradiction when considered with the rules of transfinite arithmetic. If the cardinality of the natural numbers is ∞_0, then the cardinality of the real numbers is 2^∞. In other words, $2 \bullet 2 \bullet 2 \bullet \ldots \bullet 2 = 2^\infty$, with $2 < 2^\infty$. However, transfinite arithmetic declares the following equation true:

$\infty_0 \bullet \infty_0 \bullet \infty_0 \bullet \ldots \bullet \infty_0 = \infty_0$

If we consider the previous finding ($2 \bullet 2 \bullet 2 \bullet \ldots \bullet 2 = 2^\infty$) and accept Cantor's claim that $\infty_0 < 2^\infty$, then we are stuck with the strange conclusion that $\infty_0 < 2$, which is clearly false.

No matter how many times we multiple ∞_0 times itself, the cardinality of the product stays the same (∞_0). However, in the case of infinite binary decimals, the cardinality of the product of 2s grows, to such an extent that the final result (2^∞) is larger than the result of multiplying an infinite set of ∞_0. This is not paradoxical or proof that the human mind is unable to grasp infinity; it is proof of a contradiction demonstrated with mathematical rigor. The argument could be made that Cantor never claimed that 2^∞ is the product per se of multiplying an infinite set of 2s, but his theory hinges on it because he claims 2^∞ is the cardinality of the set of infinite binary decimals (real numbers). Cantor's unusual methodology of representing the real numbers as binary decimals (as opposed to the obvious choice of base-10 decimals) allowed Cantor to derive or generate the real numbers from the natural numbers (via the power set of the natural numbers), which is required for his theory of infinite sets. It also allowed Cantor to define the cardinality of the real numbers (2^∞) and claim that this was larger than the cardinality of the natural numbers, which is circular reasoning that is merely assumed to be true with the Axiom of Power Set. However, Cantor accomplished this unusual finding at the cost of generating a contradiction.

Addendum

Continuum Hypothesis

Set theory is unable to prove or disprove the existence of an infinite set with cardinality between ∞_0 and 2^∞. One explanation for this problem is that the first cardinality (∞_0) was created ex nihilo with the Axiom of Infinity and the second cardinality (2^∞) was created ex nihilo with the Axiom of Power Set, such that the two infinite sets would not exist without the axioms. These two sets are baked into the set theory cake and cannot generate new infinite sets with different cardinalities. As such, the "discovery" of an infinite set with a new cardinality would require a new axiom.

Diagonal Argument

The diagonal argument claims we can never make a complete list of the real numbers because the diagonal number always generates a new number that is not on the list. (This same thing also applies to rational numbers expressed as decimals.) In other words, even though set theory defines the real numbers as a complete set, no matter how long the list is we can always find

RECEIVED: *Sep 06, 2019*
ACCEPTED: *Sep 29, 2019*
PUBLISHED: *Sep 31, 2019*

Online Mathematics
R.N OMJV1I4Patton
ISSN: 2672-7501
http://www. onlinemathematics.net

another number to add to the list. This is the definition of potential infinity, not a paradoxical property of infinite sets.

Hilbert's Hotel

This hotel is a special venue with infinite rooms. Even if all the rooms are "full," we are told—perhaps the natural numbers are on holiday—we can always make room for more numbers, such as the negative integers. In other words, the hotel can always accommodate more guests, unless the real numbers arrive unannounced, I suppose. Once again, this interesting property of the hotel is consistent with the definition of potential infinity and raises doubts about the idea of an infinite hotel being "full." Rather than attribute paradoxical properties to infinite sets we can use Ockham's razor to conclude that infinite sets do not exist. In which case, we could change the name to Aristotle's Hotel.

Content of the Continuum

Cantor attempted to avoid the paradoxes of the geometric continuum by defining the "arithmetic continuum," but the paradoxes remain and should be of interest to mathematicians. If we plot points on a finite continuum of arbitrary length 1, we have dimensionless points separated by finite ranges. If we increase the quantity of evenly distributed points (10, 100, 1,000, for example), the finite ranges separating the points shrink proportionally (1/10, 1/100, 1/1,000, and so on) but the finite ranges always account for 100% of the content of the continuum because points are dimensionless. We can imagine this process going to infinity, such that the limit of the finite ranges go to 0, in which case the continuum purportedly consists only of points without finite ranges. This is the problem of infinitesimals, which has no logical or metaphysical solution, aside from rejecting infinite sets. If the finite continuum consists only of points and no finite ranges, then the points are pressed together like slices of salami, touching. However, two points touching is the same as one point because points are dimensionless. Therefore, the content of the continuum is defined by finite ranges, not dimensionless points. As a result, it is a logical and metaphysical error to say that because the set of rational numbers contributes 0 to the continuum, the irrational numbers must contribute 1.

0.999… = 1

Too much ink has been spilled on this problem. The solution has nothing to do with using arithmetic or algebra to prove or disprove the equation and everything to do with infinite sets. If Aristotle is right and infinite sets to not exist, then the problem is false. If Cantor is right and infinite sets exist, then the problem is true. There is no $p/q = 0.999\ldots$, but the iterative process in problem 2 generates 0.999… as the process goes to infinity.

The Algebra of Calculus

Newton's fluxion methodology allowed him to successfully calculate the tangents of curves (derivatives), but his methodology was rejected because he used infinitesimals, which were not deemed rigorous. To avoid the problem of dividing by 0, limits replaced infinitesimals.

RECEIVED: *Sep 06, 2019*
ACCEPTED: *Sep 29, 2019*
PUBLISHED: *Sep 31, 2019*

Online Mathematics
R.N OMJV1I4Patton
ISSN: 2672-7501
http://www. onlinemathematics.net

However, the controversy wrongly stems from the premise of beginning with the goal of using an infinite, iterative process to solve the tangent problem. Suppose, on the other hand, that we are genuinely interested in solving problems when Δx is greater than 0, that is, a general formula for calculating the slope of a line connecting two points along a curve. In the case of $f(x) = x^2$, we have the following familiar solution:

$((x + \Delta x)^2 - x^2)/\Delta x$
$(\cancel{x^2} + 2x\Delta x + \Delta x^2 - \cancel{x^2})/\Delta x$
$(2x\Delta x + \Delta x^2)/\Delta x$
$\cancel{\Delta x}(2x + \Delta x)/\cancel{\Delta x}$
$2x + \Delta x$

Therefore, suppose we want to calculate the slope of the line connecting two points on the curve when x = -1 and x = 1 (Δx = 2). This is a legitimate problem that merits a solution. In this case we have the following.

$2(-1) + 2 = 0$

This change of perspective seems less controversial, generates the same mathematical results, and is probably not the first time in history that a useful formula resulted from factoring and simplification (cancelling the Δx's to avoid dividing by 0). If, on the other hand, we wish to calculate the slope of a line at a point, when Δx = 0, we can do that as well. In other words, just because we initially think about calculating the tangent of a curve as the result of an infinite, iterative process, which appears to include division by 0, this does not mean such a process is inherent to the problem.

Conclusion

This paper addressed three important errors in Cantor's theory of infinity with mathematical rigor: 1. The one-to-one correspondence methodology, 2. The problem of "filling the gaps" of the continuum, and 3. The rules of transfinite arithmetic. There are several other problems, addressed in the addendum, but these three problems should address the concerns of mathematicians. Each problem is sufficient to cast serious doubt on Cantor's system, but taken together with the addendum items pose insurmountable problems. When properly understood, the concept of potential infinity avoids all of these problems and never limits our ability to solve mathematical problems.

Acknowledgements

The author thanks the reviewers for their useful comments, which led to the improvement of the original manuscript.

Competing interests

The author declares that he has no competing interests.

RECEIVED: *Sep 06, 2019*
ACCEPTED: *Sep 29, 2019*
PUBLISHED: *Sep 31, 2019*

Online Mathematics
R.N OMJV1I4Patton
ISSN: 2672-7501
http://www. onlinemathematics.net

References:

[1] Patton, Anthony C. The End of Infinity: Where Mathematics and Philosophy Meet, (New York: Algora Publishing, 2018).

[2] Alexander, Amir, Infinitesimal: How a Dangerous Mathematical Theory Shaped the Modern World (New York: Scientific American/Farrar, Straus and Giroux, 2014).

[3] Aristotle, The Basic Works of Aristotle, edited by Richard McKeon (New York: Random House, 1941).

[4] Bagaria, Joan, “Set Theory”, The Stanford Encyclopedia of Philosophy (Winter 2014 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/win2014/entries/set-theory/>.

[5] Balaguer, Mark, “Fictionalism in the Philosophy of Mathematics", The Stanford Encyclopedia of Philosophy (Summer 2015 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/sum2015/entries/fictionalism-mathematics/>.

[6] Bell, John L., “Continuity and Infinitesimals”, The Stanford Encyclopedia of Philosophy (Winter 2014 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/win2014/entries/continuity/>.

[7] Bridges, Douglas and Palmgren, Erik, "Constructive Mathematics", The Stanford Encyclopedia of Philosophy (Winter 2013 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/win2013/entries/mathematics-constructive/>.

[8] Bueno, Otávio, "Nominalism in the Philosophy of Mathematics", The Stanford Encyclopedia of Philosophy (Spring 2014 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/spr2014/entries/nominalism-mathematics/>.

[9] Colyvan, Mark, "Indispensability Arguments in the Philosophy of Mathematics", The Stanford Encyclopedia of Philosophy (Spring 2015 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/spr2015/entries/mathphil-indis/>.

[10] Copleston, Frederick, A History of Philosophy Volume I: Greece and Rome (New York: Image Books, 1993).

[11] Dauben, Joseph Warren, “Georg Cantor: His Mathematics and Philosophy of the Infinite” (Cambridge: Harvard University Press, 1979).

[12] Franklin, James, An Aristotelian Realist Philosophy of Mathematics: Mathematics as the Science of Quantity and Structure (New York: Palgrave Macmillin, 2014).

[13] George, Alexander; Velleman, Daniel J., Philosophies of Mathematics (Malden, MA: Blackwell, 2002).

[14] Havil, Julian, The Irrationals: A Story of the Numbers You Can’t Count On (Princeton: Princeton University Press, 2012).

[15] Hume, David, A Treatise of Human Nature (New York: Barnes & Noble, 2005).

[16] Iemhoff, Rosalie, "Intuitionism in the Philosophy of Mathematics", The Stanford Encyclopedia of Philosophy (Spring 2015 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/spr2015/entries/intuitionism/>.

[17] Linnebo, Øystein, "Platonism in the Philosophy of Mathematics", The Stanford Encyclopedia of Philosophy (Winter 2013 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/win2013/entries/platonism-mathematics/>.

[18] Mancosu, Paolo, "Explanation in Mathematics", The Stanford Encyclopedia of Philosophy (Summer 2015 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/sum2015/entries/mathematics-explanation/>.

[19] Mendell, Henry, "Aristotle and Mathematics", The Stanford Encyclopedia of Philosophy (Winter 2008 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/win2008/entries/aristotle-mathematics/>.

[20] Merzbach, Uta C.; Boyer, Carl B., A History of Mathematics, third edition (Hoboken, NJ: John Wiley & Sons, 2011).

[21] Núñez, Rafael E., “Creating mathematical infinities: Metaphor, blending, and the beauty of transfinite cardinals” (Journal of Pragmatics, 37 (2005), 1717 – 1741).

RECEIVED: Sep 06, 2019
ACCEPTED: Sep 29, 2019
PUBLISHED: Sep 31, 2019

Online Mathematics
R.N OMJV1I4Patton
ISSN: 2672-7501
http://www. onlinemathematics.net

[22] Paseau, Alexander, "Naturalism in the Philosophy of Mathematics", The Stanford Encyclopedia of Philosophy (Summer 2013 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/sum2013/entries/naturalism-mathematics/>.

[23] Plato, Complete Works, edited by John M. Cooper (Indianapolis: Hackett Publishing, 1997).

[24] Reichenbach, Bruce, "Cosmological Argument", The Stanford Encyclopedia of Philosophy (Winter 2016 Edition), Edward N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/win2016/entries/cosmological-argument/>.

[25] Rodych, Victor, "Wittgenstein's Philosophy of Mathematics", The Stanford Encyclopedia of Philosophy (Summer 2011 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/sum2011/entries/wittgenstein-mathematics/>.

[26] Ross, Sir David, Aristotle, sixth edition (New York: Routledge, 1995).

[27] Russell, Bertrand, The History of Western Philosophy (New York: Simon & Schuster, 1972).

[28] —Introduction to Mathematical Philosophy (Digireads.com Publishing, 2010).

[29] Schopenhauer, Arthur, On the Fourfold Root of the Principle of Sufficient Reason (La Salle, IL: Open Court, 1999).

[30] —The World as Will and Representation (New York: Dover Publications, 1969).

[31] Shabel, Lisa, "Kant's Philosophy of Mathematics", The Stanford Encyclopedia of Philosophy (Fall 2014 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/fall2014/entries/kant-mathematics/>.

[32] Shapiro, Steward, Thinking About Mathematics: The Philosophy of Mathematics (New York: Oxford University Press, 2011).

[33] Zenkin, Alexander A., "Logic of Actual Infinity and G. Cantor's Diagonal Proof of the Uncountability of the Continuum," The Review of Modern Logic (Volume 9, Numbers 3 & 4, December 2003 – August 2004, Issue 30).