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A One-Dimensional Parabolic Inverse Problem: Review of a Compact Scheme

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This paper reviews the article by Chao-Rong and Zhi-Zhong (2009), in which a compact difference scheme for a one-dimensional parabolic inverse problem was developed and some numerical results were investigated. Consider the following differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + p(t)\,u + \phi(x,t)\,, \qquad 0 < x < 1 \text{ and } 0 < t \leqslant T$$

The above equation is an inverse parabolic problem where the boundary conditions are

$$u(0,t) = g_0(t)\,, u(1,t) = g_1(t)\,, \qquad 0 \leqslant t \leqslant T$$

And the initial conditions are specified as

$$u(x,0) = f(x)\,, \qquad 0 < x < 1$$

Moreover, assume an extra specification given at the specific point $x*$, with

$$u(x^*,t) = E(t)\,, \qquad 0 \leqslant t \leqslant T$$

Here $E(t)\,, f(x)\,, g_0(t)\,, g_1(t)\,, and\ \phi(x,t)$ are known real functions while $u(x,t)$ and $p(t)$ are unknown.

Furthermore, let

$x^* \in (0,1)$

$|E(t)| \geqslant E_0 > 0$

An implicit difference scheme with convergence order of $O(\tau + h^2)$ for $u(x,t)$ and $\tau = O(h)$ for $p(t)$ had been developed by Cannon et al. [1]. Using the following over-specified integrals

$$\int_0^1 k(x)\, u(x,t)\, \mathrm{d}x = E(t)\,, 0 \leqslant t \leqslant T$$

$$\int_0^{s(t)} k(x)\, u(x,t)\, \mathrm{d}x = E(t)\,, 0 \leqslant t \leqslant T, 0 < s(t) < 1$$

Dehghan [2] developed 4 numerical schemes (Crandall's scheme, three-point backward scheme, five-point forward scheme, and three-point forward scheme) to solve the above inverse parabolic problem.

Furthermore, some numerical methods were developed by Daoud and Subasi [3] for the 2-dimensional case of the above problem.

More specifically, consider the following meshes,

$\Omega_h = \{x_i | x_i = ih, 0 \leqslant i \leqslant M\}$

$\Omega_\tau = \{t_n | t_n = n\tau, 0 \leqslant n \leqslant N\}$

And let $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$ be the two dimensional mesh, and take an integer number k_0 so that the point of over-specification would be $x^* = x_{k_0}$.

Now on $\Omega_{h\tau}$ consider the functions

$\Phi_i^n = \phi(x_i,t_n)\,,E^n = E(t_n)\,,(E')^n = E'(t_n)\,,0 \leqslant i \leqslant M,0 \leqslant n \leqslant N.$

For the above inverse parabolic problem, a backward Crandall's scheme can be developed as follows;

$$\frac{1}{12}(\delta_t u_{i-1}^{n+\frac{1}{2}} + 10\delta_t u_i^{n+\frac{1}{2}} + \delta_t u_{i+1}^{n+\frac{1}{2}}) = \delta_x^2 u_i^{n+\frac{1}{2}} + p^{n+1}u_i^n + \Phi_i^{n+1},1 \leqslant i \leqslant M-1,0 \leqslant n \leqslant N-1$$

$$p^{n+1} = \frac{1}{E^{n+1}}[(E')^{n+1} - \frac{1}{12h^2}(-u_{k_0-2}^{n+1} + 16u_{k_0-1}^{n+1} - 30u_{k_0}^{n+1} + 16u_{k_0+1}^{n+1} - u_{k_0+2}^{n+1}) - \Phi_{k_0}^{n+1}],0 \leqslant n \leqslant N-1$$

Where $p^0 = \frac{1}{f(x^*)}[(E')^0 - f''(x^*) - \Phi_{k_0}^0]$ and $u_i^0 = f(x_i)\,,1 \leqslant i \leqslant M-1$ and

$$u_i^0 = f(x_i)\,,1 \leqslant i \leqslant M-1$$

$$u_0^n = g_0(t_n)\,,u_M^n = g_1(t_n)\,,0 \leqslant n \leqslant N$$

Note that here, the following notations are used:

$$u_i^{n+\frac{1}{2}} = \frac{u_i^{n+1} + u_i^n}{2},\delta_t u_i^{n+\frac{1}{2}} = \frac{u_i^{n+1} - u_i^n}{\tau},D_{\hat{t}} u_i^n = \frac{u_i^{n+1} - u_i^{n-1}}{2\tau},$$
$$\bar{u}_i^n = \frac{u_i^{n+1} + u_i^{n-1}}{2},\delta_x^2 u_i^n = \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{h^2},$$
$$\|u^n\| = [h\sum_{i=1}^{M-1}(u_i^n)^2]^{1/2},\|u^n\|_\infty = \max_{0\leqslant i\leqslant M}|u_i^n|$$

Clearly the above system constitutes a linear system with the unknowns $(u_1^{n+1},u_2^{n+1},\dots,u_{M-1}^{n+1},p^{n+1})^{\mathrm{T}}$ and can be solved in $\mathrm{O}(\tau)$.

To developing a compact scheme for the above inverse parabolic problem, consider

$$v = \frac{\partial^2 u}{\partial x^2}$$

If we substitute it into the differential equation, we obtain

$$v = \frac{\partial u}{\partial t} - p(t)\,u - \phi(x,t)$$

Now let $U_i^n = u(x_i,t_n)$,$V_i^n = v(x_i,t_n)$,$P^n = p(t_n)$ be defined on $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$ for $0 \leqslant i \leqslant M$, $0 \leqslant n \leqslant N$. Then by substituting them into the PDE we get

$$v(x_i,t_n) = \frac{\partial u}{\partial t}(x_i,t_n) - p(t_n)\,u(x_i,t_n) - \phi(x_i,t_n)$$

Which can also be written as:

$$V_i^n = \frac{\partial u}{\partial t}(x_i,t_n) - P^n U_i^n - \Phi_i^n$$

Taylor expansion can be applied to the above equation to obtain

$$\frac{1}{12}[\frac{\partial u}{\partial t}(x_{i-1},t_n) + 10\frac{\partial u}{\partial t}(x_i,t_n) + \frac{\partial u}{\partial t}(x_{i+1},t_n)]$$
$$= \delta_x^2 U_i^n + \frac{1}{12}P^n(U_{i-1}^n + 10U_i^n + U_{i+1}^n) + \frac{1}{12}(\Phi_{i-1}^n + 10\Phi_i^n + \Phi_{i+1}^n) + O(h^4)$$

Employing yet another Taylor expansion to the above equation yields

$$\frac{1}{12}(D_{\hat{t}}U_{i-1}^n + 10D_{\hat{t}}U_i^n + D_{\hat{t}}U_{i+1}^n)$$
$$= \delta_x^2 \bar{U}_i^n + \frac{1}{12}P^n(\bar{U}_{i-1}^n + 10\bar{U}_i^n + \bar{U}_{i+1}^n) + \frac{1}{12}(\Phi_{i-1}^n + 10\Phi_i^n + \Phi_{i+1}^n) + (e_1)_i^n$$

Note that here a constant m could always be found to satisfy $|\ (e_1)_i^n\ | \leqslant m(\tau^2 + h^4)$.

Now if substitute above Taylor expansion into the parabolic PDF and rearrange the terms, we get

$$p(t_n) = \frac{1}{E^n}[(E')^n - \frac{\partial^2 u}{\partial x^2}(x_{k_0},t_n) - \Phi_{k_0}^n], \quad 1 \leqslant n \leqslant N$$

Now we can substitute $p(t_n)$ back into the PDE and obtain

$$\frac{\partial^2 u}{\partial x^2}(x_{k_0},t_n) = \frac{1}{12h^2}(-U_{k_0-2}^n + 16U_{k_0-1}^n - 30U_{k_0}^n + 16U_{k_0+1}^n - U_{k_0+2}^n) + O(h^4)$$

Note that evidently, a constant m' can be found to satisfy $|\ (e_2)^n\ | \leqslant m'h^4$ for $1 \leqslant n \leqslant N$.

Here we can apply the Taylor expansion again and obtain

$U_i^1 = f(x_i) + \tau[f''(x_i) + P^0 f(x_i) + \Phi_i^0] + (e_3)_i$ and we can always find $m"$ such that

$|\ (e_3)_i\ | \leqslant m"\ \tau^2$.

Now using the initial values, we can write

$$P^0 = \frac{1}{f(x^*)}[(E')^0 - f''(x^*) - \Phi_{k_0}^0]$$

$$U_i^0 = f(x_i)\,, \quad 1 \leqslant i \leqslant M - 1$$

And $\quad U_0^n = g_0(t_n)\,, U_M^n = g_1(t_n)\,, \ \ 0 \leqslant n \leqslant N.$

Higher order terms can be eliminated from the above Taylor expansions, and putting them altogether we can obtain the following 3 level linear scheme:

$$\frac{1}{12}(D_{\hat{t}}u_{i-1}^{n}+10D_{\hat{t}}u_{i}^{n}+D_{\hat{t}}u_{i+1}^{n})$$
$$=\delta_x^2\bar{u}_i^n+\frac{1}{12}p^n(\bar{u}_{i-1}^n+10\bar{u}_i^n+\bar{u}_{i+1}^n)+\frac{1}{12}(\Phi_{i-1}^n+10\Phi_i^n+\Phi_{i+1}^n),1\leqslant i$$
$$\leqslant M-1,1\leqslant n\leqslant N-1,$$

$$p^n=\frac{1}{E^n}[(E')^n-\frac{1}{12h^2}(-u_{k_0-2}^n+16u_{k_0-1}^n-30u_{k_0}^n+16u_{k_0+1}^n-u_{k_0+2}^n)-\Phi_{k_0}^n],1\leqslant n\leqslant N,$$

$$u_i^1=f(x_i)+\tau[f''(x_i)+p^0f(x_i)+\Phi_i^0],1\leqslant i\leqslant M-1,$$

$$u_i^0=f(x_i),1\leqslant i\leqslant M-1,$$

$$u_0^n=g_0(t_n),u_M^n=g_1(t_n),0\leqslant n\leqslant N.$$

$$p^0=\frac{1}{f(x^*)}[(E')^0-f''(x^*)-\Phi_{k_0}^0],$$

Using the boundary value conditions, we can evidently specify $\{u_i^0,u_i^1 \mid 0\leqslant i\leqslant M\}\cup\{p^0,p^1\}$ and if we also specify $\{u_i^{n-1},u_i^n \mid 0\leqslant i\leqslant M\}\cup\{p^n\}$, then we can write the above linearized difference scheme as

$$\frac{1}{12}(D_{\hat{t}}u_{i-1}^{n}+10D_{\hat{t}}u_{i}^{n}+D_{\hat{t}}u_{i+1}^{n})$$
$$=\delta_x^2\bar{u}_i^n+\frac{1}{12}p^n(\bar{u}_{i-1}^n+10\bar{u}_i^n+\bar{u}_{i+1}^n)+\frac{1}{12}(\Phi_{i-1}^n+10\Phi_i^n+\Phi_{i+1}^n),1\leqslant i$$
$$\leqslant M-1,$$

Where $u_0^{n+1}=g_0(t_{n+1}),u_M^{n+1}=g_1(t_{n+1})$.

This forms a tridiagonal linear system that can be solved to get $\{u_i^{n+1} \mid 0 \leqslant i \leqslant M\}$ and consequently obtain p^{n+1} by using

$$p^n = \frac{1}{E^n}[(E')^n - \frac{1}{12h^2}(-u_{k_0-2}^n + 16u_{k_0-1}^n - 30u_{k_0}^n + 16u_{k_0+1}^n - u_{k_0+2}^n) - \Phi_{k_0}^n]$$

In the following section we will prove that the above difference scheme is well-posed and has a unique solution. To do so we will need the following theorem:

Lemma.

Consider the following mesh $\Omega_h = \{x_i | x_i = ih, 0 \leqslant i \leqslant M, Mh = 1\}$ and function $g = \{g_i | 0 \leqslant i \leqslant M\}$ such that $g_0 = 0, g_M = 0$. Then we can write

$$h\sum_{i=1}^{M-1} g_i^2 \leqslant \frac{1}{6}h\sum_{i=1}^{M}(\frac{g_i - g_{i-1}}{h})^2$$

Theorem.

Assume that $\{u_i^{n-1}, u_i^n | 0 \leqslant i \leqslant M\} \cup \{p^n\}$ is specified. Then if

$p^n \leqslant 6$ or

$p^n > 6$ and $\tau < 1/(p^n - 6)$

the above difference scheme uniquely specifies $\{u_i^{n+1} | 0 \leqslant i \leqslant M\} \cup \{p^{n+1}\}$.

proof.

Evidently, we can write the above difference scheme as

$$\frac{1}{24\tau}(u_{i-1}^{n+1}+10u_i^{n+1}+u_{i+1}^{n+1})=\frac{1}{2}\delta_x^2u_i^{n+1}+\frac{1}{24}p^n(u_{i-1}^{n+1}+10u_i^{n+1}+u_{i+1}^{n+1}),1\leqslant i\leqslant M-1$$

Now let's multiply both sides by $2hu_i^{n+1}$ and then add all linear equations from 1 to M – 1. We will obtain

$$\frac{1}{12\tau}h\sum_{i=1}^{M-1}(u_{i-1}^{n+1}+10u_i^{n+1}+u_{i+1}^{n+1})u_i^{n+1}$$

$$=h\sum_{i=1}^{M-1}(\delta_x^2u_i^{n+1})u_i^{n+1}+\frac{1}{12}p^nh\sum_{i=1}^{M-1}(u_{i-1}^{n+1}+10u_i^{n+1}+u_{i+1}^{n+1})u_i^{n+1}$$

Now we will apply the boundary conditions $u_0^{n+1}=0,u_M^{n+1}=0$ to the above equation and obtain

$$\frac{1}{12\tau}h\sum_{i=1}^{M-1}(u_{i-1}^{n+1}+10u_i^{n+1}+u_{i+1}^{n+1})u_i^{n+1}$$

$$=-h\sum_{i=1}^{M-1}(\delta_xu_{i-\frac{1}{2}}^{n+1})^2+\frac{1}{12}p^nh\sum_{i=1}^{M-1}(u_{i-1}^{n+1}+10u_i^{n+1}+u_{i+1}^{n+1})u_i^{n+1}$$

Now if we apply the lemma to the above equation and simplify the terms we get

$$\frac{1}{12}(\frac{1}{\tau}-p^n)h\sum_{i=1}^{M-1}(u_{i-1}^{n+1}+10u_i^{n+1}+u_{i+1}^{n+1})u_i^{n+1}+6\|u^{n+1}\|^2\leqslant 0$$

Moreover, we can write

$$\frac{2}{3}\|u^{n+1}\|^2 \leqslant \frac{1}{12}h\sum_{i=1}^{M-1}(u_{i-1}^{n+1}+10u_i^{n+1}+u_{i+1}^{n+1})u_i^{n+1} \leqslant \|u^{n+1}\|^2$$

And for $\frac{1}{\tau} - p^n \geqslant 0$ we have $\frac{2}{3}(\frac{1}{\tau} - p^n)\|u^{n+1}\|^2 + 6\|u^{n+1}\|^2 \leqslant 0$.

Hence when $\frac{1}{\tau} - p^n \geqslant 0$ we have

$\|u^{n+1}\| = 0$

On the other hand, when $\frac{1}{\tau} - p^n < 0$ we can write

$6\|u^{n+1}\|^2 \leqslant (p^n - \frac{1}{\tau})\|u^{n+1}\|^2$ and obtain

$(6 - p^n + \frac{1}{\tau})\|u^{n+1}\|^2 \leqslant 0$

Hence it is easy to see that when

$p^n \leqslant 6$ or

$p^n > 6$ and $\tau < 1/(p^n - 6)$

We have $\|u^{n+1}\| = 0$.

So $\|u^{n+1}\| = 0$ holds in both cases and hence the linear system produced by the difference scheme always has a unique solution.

Here we will finish the review of the article with a numerical example that employs that above linearized difference scheme on the following parabolic inverse problem:

Let

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + p(t)\,u + [\pi^2 - (t+1)^2]\mathrm{e}^{-t^2}[\cos(\pi x) + \sin(\pi x)], 0 < x < 1, 0 < t \leqslant 1$$

And consider the following boundary value conditions:

$u(x,0) = \cos(\pi x) + \sin(\pi x)$,$0 < x < 1$

$u(0,t) = \mathrm{e}^{-t^2}, u(1,t) = -\mathrm{e}^{-t^2}, 0 \leqslant t \leqslant 1$

$u(0.25,t) = \sqrt{2}\mathrm{e}^{-t^2}, 0 \leqslant t \leqslant 1$

It is easy to check that the above differential equation with the specified boundary conditions has the following exact solution

$u(x,t) = \mathrm{e}^{-t^2}[\cos(\pi x) + \sin(\pi x)]$ and $p(t) = 1 + t^2$.

Here we will use the infinity norm to define the following errors

$E_\infty(h,\tau) = \max_{0\leqslant n\leqslant N}\{\max_{0\leqslant i\leqslant M}|U_i^n - u_i^n|\}$

$F_\infty(h,\tau) = \max_{0\leqslant n\leqslant N}|P^n - p^n|$

Using the developed difference scheme and solving the obtained linear system, the following results were obtained.

$\boldsymbol{M}$	$\boldsymbol{N}$	$\boldsymbol{E_\infty(h,\tau)}$	$\boldsymbol{E_\infty(2h,4\tau)/E_\infty(h,\tau)}$	$\boldsymbol{F_\infty(h,\tau)}$	$\boldsymbol{F_\infty(2h,4\tau)/F_\infty(h,\tau)}$
40	80	3.001612e−3	*	7.537490e−2	*
80	320	1.847058e−4	16.250	4.638067e−3	16.251
160	1280	1.153267e−5	16.015	2.895918e−4	16.015
320	5120	7.207327e−7	16.001	1.809814e−5	16.001

Here it can be clearly seen that when reducing time mesh by a factor of 4 and spatial mesh by a factor of 2, results in a reduction in approximation error of factor 16.

To compare these results with the implicit Crandall's scheme developed by Dehghan [2], let $p^{n+1(0)} = p^n$ and put in into

$$\frac{1}{12}(\delta_t u_{i-1}^{n+\frac{1}{2}} + 10\delta_t u_i^{n+\frac{1}{2}} + \delta_t u_{i+1}^{n+\frac{1}{2}}) = \delta_x^2 u_i^{n+\frac{1}{2}} + p^{n+1}u_i^n + \Phi_i^{n+1}, 1 \leqslant i \leqslant M-1, 0 \leqslant n \leqslant N-1$$

Then we can obtain $\{u_i^{n+1(0)} 1 \leqslant i \leqslant M-1\}$.

Hence by using

$$p^{n+1} = \frac{1}{E^{n+1}}[(E')^{n+1} - \frac{1}{12h^2}(-u_{k_0-2}^{n+1} + 16u_{k_0-1}^{n+1} - 30u_{k_0}^{n+1} + 16u_{k_0+1}^{n+1} - u_{k_0+2}^{n+1}) - \Phi_{k_0}^{n+1}], 0 \leqslant n \leqslant N-1$$

We can calculate $p^{n+1(1)}$. This procedure can be repeated until $\left|p^{n+1(l)} - p^{n+1(l-1)}\right| < 10^{-5}$.

Then we can let $u_i^{n+1} = u_i^{n+1(l)}, 1 \leqslant i \leqslant M-1$ and calculate $p^{n+1} = p^{n+1(l)}$.

M	N	$E_\infty(h,\tau)$	$E_\infty(2h,4\tau)/E_\infty(h,\tau)$	$F_\infty(h,\tau)$	$F_\infty(2h,4\tau)/F_\infty(h,\tau)$
40	80	8.098521e−2	*	2.082863	*
80	320	2.378824e−2	3.4044	6.086413e−1	3.4221
160	1280	6.191500e−3	3.8420	1.581301e−1	3.8489
320	5120	1.563503e−3	3.9600	3.991209e−2	3.9619

The above table is obtained by using Dehghan’s difference scheme and as can be seen the compact scheme has significantly lower computational costs and higher accuracy.

References

[1] J.R. Cannon, Y.L. Lin, S. Xu, Numerical procedures for the determination of an unknown coefficient in semi-linear parabolic differential equations, Inverse Probl., 10 (1994), pp. 227-243

[2] M. Dehghan, Finding a control parameter in one-dimensional parabolic equations, Appl. Math. Comput., 135 (2003), pp. 491-503

[3] D.S. Daoud, D. Subasi, A splitting up algorithm for the determination of the control parameter in multidimensional parabolic problem, Appl. Math. Comput., 166 (2005), pp. 584-595