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RESEARCH ARTICLE

A Certain Subclass of Uniformly Convex Functions with Negative Coefficients Defined by Gegenbauer Polynomials

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Abstract

In this paper, we introduce a new subclass of uniformly convex functions defined by Gegenbauer polynomials with negative coefficients. For functions in the class TS, we attain coefficient bounds, growth distortion properties, extreme points and radii of close-to-convexity, starlikeness and convexity. For this class, we also produced modified Hadamard product, convolution, and integral operators.

Keywords: analytic, coefficient bounds, extreme points, convolution, polynomials.

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1 Introduction

Let A indicate the class of all functions $u(z)$ of the form

$$u(z) = z + \sum_{\eta=2}^{\infty} a_\eta z^\eta \tag{1}$$

in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. Let S be a subclass of A that contains univalent functions and satisfies the usual normalization condition $u(0) = u'(0) - 1 = 0$. The subset of A comprising of functions $u(z)$ that are all univalent in E is represented by S. A function $u \in A$ is a starlike function of the order $\upsilon, 0 \le \upsilon < 1$, if it fulfils

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \upsilon, (z \in E). \tag{2}$$

We indicate this class with $S^*(\upsilon)$.

A function $u \in A$ is a convex function of the order $\upsilon, 0 \le \upsilon < 1$, if it fulfils

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \upsilon, (z \in E). \tag{3}$$

We indicate this class with $K(\upsilon)$.

The regular classes of starlike and convex functions in E are $S^*(0) = S^*$ and $K(0) = K$, respectively.

Let T indicate the class of functions analytic in E that are of the form

$$u(z) = z - \sum_{\eta=2}^{\infty} a_\eta z^\eta, \quad (a_\eta \ge 0,\ z \in E) \tag{4}$$

and let $T^*(\upsilon) = T \cap S^*(\upsilon)$, $C(\upsilon) = T \cap K(\upsilon)$. Silverman [21] has thoroughly studied the class $T^*(\upsilon)$ and related classes, which have some interesting properties. In [2,5], and others have recently looked into several subclasses of T.

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For $u \in A$ given by (1) and $g(z)$ given by

$$g(z) = z + \sum_{\eta=2}^{\infty} b_\eta z^\eta$$

their convolution indicate by $(u * g)$, is specified as

$$(u*g)(z) = z + \sum_{\eta=2}^{\infty} a_\eta b_\eta z^\eta = (g*u)(z), \quad (z \in E).$$

Note that $u * g \in A$.

The following subclasses were introduced and examined by Goodman [8,9] and Ronning [15,16] :

(1). If a function $u \in A$ satisfies the condition, it is said to be in the class $UCV(\rho,\gamma)$, a uniformly $\gamma-$ convex function

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)} - \rho\right\} > \gamma\left|\frac{zu''(z)}{u'(z)}\right|, \tag{5}$$

where $\gamma \geq 0, -1 < \rho \leq 1$ and $\rho + \gamma \geq 0$.

(2). If a function $u \in A$ satisfies the condition, it is said to be in the class $SP(\rho,\gamma)$, uniformly $\gamma-$ starlike function.

$$\Re\left\{\frac{zu'(z)}{u(z)} - \rho\right\} > \gamma\left|\frac{zu'(z)}{u(z)} - 1\right|, \tag{6}$$

where $\gamma \geq 0, -1 < \rho \leq 1$ and $\rho + \gamma \geq 0$.

Indeed it follows from (5) and (6) that

$$u \in UCV(\rho,\gamma) \Leftrightarrow zu' \in SP(\rho,\gamma). \tag{7}$$

For $\gamma = 0$, the classes $K(0) = K$ and $S*(0) = S*$ are provided, respectively. Uniformly convex functions are functions of the class $UCV(0,1) \equiv UCV$ that were introduced by Goodman with geometric interpretation in [8]. Ronning describes the class $SP(0,1) \equiv SP$ in [15]. For $\rho = 0$, the class $UCV(0,\gamma) \equiv \gamma - UCV$ and $SP(0,\gamma) \equiv \gamma - SP$ are defined respectively, by Kanas and Wisniowska in [10,11].

Further, Murugusundarmoorthy and Magesh [13], Santosh et al. [18], Thirupathi Reddy and Venkateswarlu [25] and also see ([1,3,7,12,17,23]) have also studied interesting properties for the classes $UCV(\rho,\gamma)$ and $SP(\rho,\gamma)$.

Szynal [24] introduced class $\mathcal{T}(\lambda), \lambda \geq 0$ and examined it as the subclass of $\mathcal{A}$ consisting of functions of the form

$$u(z) = \int_{-1}^{1} k(z,t)d\mu(t), \tag{8}$$

where

$$k(z,t) = \frac{z}{(1-2tz+z^2)^\lambda}, \quad (z \in U, t \in [-1,1]) \tag{9}$$

and μ is a probability measure on the interval $[-1,1]$. The collection of such measures on $[a,b]$ is denoted by $P_{[a,b]}$.

In (9), the Taylor series expansion of the function gives

$$k(z,t) = z + c_1^\lambda(t)z^2 + c_2^\lambda(t)z^3 + \cdots \tag{10}$$

and the coefficients for (10) were given below:

$$\begin{aligned} &c_0^\lambda(t) = 1;\ c_1^\lambda(t) = 2\lambda t; c_2^\lambda(t) = 2\lambda(\lambda+1)t^2 - \lambda; \\ &c_3^\lambda(t) = \frac{4}{3}\lambda(\lambda+1)(\lambda+2)t^3 - 2\lambda(\lambda+1)t \ \ \cdots \end{aligned} \tag{11}$$

where $c_\eta^\lambda(t)$ indicates the Gegenbauer polynomial of degree η. Varying the parameter λ in (10), we obtain the class of typically real functions studied by [6,14,20] and [22].

Let $\mathcal{G}_{\lambda,t}$:$A \to A$ specified in terms of convolution by

$$\mathcal{G}_{\lambda,t}u(z) = k(z,t) * u(z),$$

we have

$$\mathcal{G}_{\lambda,t}u(z)=z+\sum_{\eta=2}^{\infty}\phi(\lambda,t,\eta)a_{\eta}z^{\eta} \tag{12}$$

where $\phi(\lambda,t,\eta)=C_{\eta-1}^{\lambda}(t)$.

We can now describe a new subclass of functions belonging to the class A by using the linear operator $\mathcal{G}_{\lambda,t}$.

Definition 1.1. *For $-1\leq\upsilon<1$ and $\rho\geq0$, we let $TS(\upsilon,\rho,\lambda,t)$ be the subclass of A consisting of functions of the form (4) and fulfiling the analytic condition*

$$\Re\left\{\frac{z(\mathcal{G}_{\lambda,t}u(z))'}{\mathcal{G}_{\lambda,t}u(z)}-\upsilon\right\}\geq\rho\left|\frac{z(\mathcal{G}_{\lambda,t}u(z))'}{\mathcal{G}_{\lambda,t}u(z)}-1\right|, \tag{13}$$

for $z\in E$.

The class $TS(\upsilon,\rho,\lambda,t)$ can be reduced to the class studied earlier by Ronning [15,16] by suitably specialising the values of υ and ρ. The primary aim of this paper is to examine some common geometric function theory properties such as coefficient bounds, distortion properties, extreme points, radii of starlikeness and convexity, Hadamard product, and convolution and integral operators for the class.

2 Coefficient Bounds

We get a required and adequate condition for function $u(z)$ in the class $TS(\upsilon,\rho,\lambda,t)$ in this section. To find the coefficient estimates for our class, we use the approach proposed by Aqlan et al. [4].

Theorem 2.1. *The function u defined by (4) is in the class $TS(\upsilon,\rho,\lambda,t)$ if and only if*

$$\sum_{\eta=2}^{\infty}[\eta(1+\rho)-(\upsilon+\rho)]\phi(\lambda,t,\eta)|a_{\eta}|\leq1-\upsilon, \tag{14}$$

where $-1\leq\upsilon<1,\rho\geq0$. The result is sharp.

Proof. We have $f\in TS(\upsilon,\rho,\lambda,t)$ if and only if the condition (13) satisfied. Upon the fact that

$$\Re(w)>\rho|w-1|+\upsilon\Leftrightarrow\Re\{w(1+\rho e^{i\theta})-\rho e^{i\theta}\}>\upsilon,\ -\pi\leq\theta\leq\pi.$$

Equation (13) may be written as

$$\Re\left\{\frac{z(\mathcal{G}_{\lambda,t}u(z))'}{\mathcal{G}_{\lambda,t}u(z)}(1+\rho e^{i\theta})-\rho e^{i\theta}\right\}=\Re\left\{\frac{z(\mathcal{G}_{\lambda,t}u(z))'(1+\rho e^{i\theta})-\rho e^{i\theta}\mathcal{G}_{\lambda,t}u(z)}{\mathcal{G}_{\lambda,t}u(z)}\right\}>\upsilon. \tag{15}$$

Now, we let

$$E(z)=z(\mathcal{G}_{\lambda,t}u(z))'(1+\rho e^{i\theta})-\rho e^{i\theta}\mathcal{G}_{\lambda,t}u(z)$$
$$F(z)=\mathcal{G}_{\lambda,t}u(z).$$

Then (15) is equivalent to

$$|E(z)+(1-\upsilon)F(z)|>|E(z)-(1+\upsilon)F(z)|,\ \text{for } 0\leq\upsilon<1.$$

For $E(z)$ and $F(z)$ as above, we have

$$|E(z)+(1-\upsilon)F(z)|\geq(2-\upsilon)|z|-\sum_{\eta=2}^{\infty}[\eta+1-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)|a_{\eta}||z^{\eta}|$$

and similarly

$$|E(z)-(1+\upsilon)F(z)|\leq\upsilon|z|-\sum_{\eta=2}^{\infty}[\eta-1-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)|a_{\eta}||z^{\eta}|.$$

Therefore

$$|E(z)+(1-\upsilon)F(z)|-|E(z)-(1+\upsilon)F(z)|$$
$$\geq 2(1-\upsilon)-2\sum_{\eta=2}^{\infty}[\eta-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)|a_\eta|$$

or $$\sum_{\eta=2}^{\infty}[\eta-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)|a_\eta|\leq(1-\upsilon),$$

which yields (14).
On the other hand, we must have

$$\Re\left\{\frac{z(\mathcal{G}_{\lambda,t}u(z))'}{\mathcal{G}_{\lambda,t}u(z)}(1+\rho e^{i\theta})-\rho e^{i\theta}\right\}\geq\upsilon.$$

Upon choosing the values of z on the positive real axis where $0\leq|z|=r<1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\upsilon)r-\sum\limits_{\eta=2}^{\infty}[\eta-\upsilon+\rho e^{i\theta}(\eta-1)]\phi(\lambda,t,\eta)|a_\eta|\,r^\eta}{z-\sum\limits_{\eta=2}^{\infty}\phi(\lambda,t,\eta)|a_\eta|\,r^\eta}\right\}\geq 0.$$

Since $\Re(-e^{i\theta})\geq-|e^{i\theta}|=-1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\upsilon)r-\sum\limits_{\eta=2}^{\infty}[\eta-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)|a_\eta|\,r^\eta}{z-\sum\limits_{\eta=2}^{\infty}\phi(\lambda,t,\eta)|a_\eta|\,r^\eta}\right\}\geq 0.$$

Letting $r\to 1^-$, we get the desired result. Finally the result is sharp with the extremal function u given by

$$u(z)=z-\frac{1-\upsilon}{[\eta(1+\rho)-(\upsilon+\rho)]\phi(\lambda,t,\eta)}z^\eta. \tag{16}$$

□

3 Growth and Distortion Theorems

Theorem 3.1. *Let the function u defined by (4) be in the class $TS(\upsilon,\rho,\lambda,t)$. Then for $|z|=r$*

$$r-\frac{1-\upsilon}{2\lambda t(2-\upsilon+\rho)}r^2\leq|u(z)|\leq r+\frac{1-\upsilon}{2\lambda t(2-\upsilon+\rho)}r^2. \tag{17}$$

Equality holds for the function

$$u(z)=z-\frac{1-\upsilon}{2\lambda t(2-\upsilon+\rho)}z^2. \tag{18}$$

Proof. Since the other inequality can be explained using identical reasoning, we just prove the right hand side inequality in (17). In view of Theorem 2.1, we have

$$\sum_{\eta=2}^{\infty}|a_\eta|\leq\frac{1-\upsilon}{2\lambda t(2-\upsilon+\rho)}. \tag{19}$$

Since,

$$u(z)=z-\sum_{\eta=2}^{\infty}a_\eta z^\eta$$
$$|u(z)|=\left|z-\sum_{\eta=2}^{\infty}a_\eta z^\eta\right|\leq r+\sum_{\eta=2}^{\infty}|a_\eta|r^\eta\leq r+r^2\sum_{\eta=2}^{\infty}|a_\eta|$$
$$\leq r+\sum_{\eta=2}^{\infty}\frac{1-\upsilon}{2\lambda t(2-\upsilon+\rho)}r^2$$

which yields the right hand side inequality of (17). □

Next, by using the same technique as in proof of Theorem 3.1, we give the distortion result.

Theorem 3.2. *Let the function u defined by (4) be in the class $TS(\upsilon,\rho,\lambda,t)$. Then for $|z|=r$*

$$1-\frac{(1-\upsilon)}{\lambda t(2-\upsilon+\rho)}r \le |u'(z)| \le 1+\frac{(1-\upsilon)}{\lambda t(2-\upsilon+\rho)}r.$$

Equality holds for the function given by (18).

Proof. Since $f \in TS(\upsilon,\rho,\lambda,t)$ by Theorem 2.1, we have that

$$2\lambda t\,[2(1+\rho)-(\upsilon+\rho)]\sum_{\eta=2}^{\infty}\eta a_\eta \le [\eta(1+\rho)-(\upsilon+\rho)]\,\phi(\lambda,t,\eta)|a_\eta| \le 1-\upsilon$$

or

$$\sum_{\eta=2}^{\infty}\eta|a_\eta| \le \frac{(1-\upsilon)}{\lambda t(2-\upsilon+\rho)}\;.$$

Thus from (19), we obtain

$$\begin{aligned}|u'(z)| &\le 1+r\sum_{\eta=2}^{\infty}\eta|a_\eta|\\ &\le 1+\frac{(1-\upsilon)}{\lambda t(2-\upsilon+\rho)}r\end{aligned}$$

which is right hand inequality of Theorem 3.2.

On the other hand, similarly

$$|u'(z)| \ge 1-\frac{(1-\upsilon)}{\lambda t(2-\upsilon+\rho)}r$$

and thus proof is completed. □

Theorem 3.3. *If $u \in TS(\upsilon,\rho,\lambda,t)$ then $u \in TS(\gamma)$, where*

$$\gamma = 1-\frac{(\eta-1)(1-\upsilon)}{[\eta-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)-(1-\upsilon)}.$$

Equality holds for the function given by (18).

Proof. It is sufficient to show that (14) implies

$$\sum_{\eta=2}^{\infty}(\eta-\gamma)|a_\eta| \le 1-\gamma,$$

that is

$$\frac{\eta-\gamma}{1-\gamma} \le \frac{[\eta-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)}{(1-\upsilon)}$$

then

$$\gamma \le 1-\frac{(\eta-1)(1-\upsilon)}{[\eta-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)-(1-\upsilon)}.$$

The above inequality holds true for $\eta \in \mathbb{N}_0, \eta \ge 2, \rho \ge 0$ and $0 \le \upsilon < 1$. □

4 Extreme Points

Theorem 4.1. *Let $u_1(z)=z$ and*

$$u_\eta(z) = z-\frac{1-\upsilon}{[\eta(\rho+1)-(\upsilon+\rho)]\phi(\lambda,t,\eta)}z^\eta, \tag{20}$$

for $\eta=2,3,\cdots$. Then $u(z)\in TS(\upsilon,\rho,\lambda,t)$ if and only if $u(z)$ can be expressed in the form $u(z)=\sum_{\eta=1}^{\infty}\zeta_\eta u_\eta(z)$, where $\zeta_\eta \ge 0$ and $\sum_{\eta=1}^{\infty}\zeta_\eta = 1$.

Proof. Suppose $u(z)$ can be expressed as in (20). Then

$$\begin{aligned} u(z) &= \sum_{\eta=1}^{\infty} \zeta_\eta u_\eta(z) = \zeta_1 u_1(z) + \sum_{\eta=2}^{\infty} \zeta_\eta u_\eta(z) \\ &= \zeta_1 u_1(z) + \sum_{\eta=2}^{\infty} \zeta_\eta \left\{ z - \frac{1-\upsilon}{[\eta(\rho+1)-(\upsilon+\rho)]\phi(\lambda,t,\eta)} z^\eta \right\} \\ &= \zeta_1 z + \sum_{\eta=2}^{\infty} \zeta_\eta z - \sum_{\eta=2}^{\infty} \zeta_\eta \left\{ \frac{1-\upsilon}{[\eta(\rho+1)-(\upsilon+\rho)]\phi(\lambda,t,\eta)} z^\eta \right\} \\ &= z - \sum_{\eta=2}^{\infty} \zeta_\eta \left\{ \frac{1-\upsilon}{[\eta(\rho+1)-(\upsilon+\rho)]\phi(\lambda,t,\eta)} z^\eta \right\}. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{\eta=2}^{\infty} \zeta_\eta \left(\frac{1-\upsilon}{[\eta(\rho+1)-(\upsilon+\rho)]\phi(\lambda,t,\eta)} \right) \left(\frac{[\eta(\rho+1)-(\upsilon+\rho)]\phi(\lambda,t,\eta)}{1-\upsilon} \right) \\ &= \sum_{\eta=2}^{\infty} \zeta_\eta = \sum_{\eta=1}^{\infty} \zeta_\eta - \zeta_1 = 1 - \zeta_1 \leq 1. \end{aligned}$$

So, by Theorem 2.1, $u \in TS(\upsilon,\rho,\lambda,t)$.

Conversely, we suppose $u \in TS(\upsilon,\rho,\lambda,t)$. Since

$$|a_\eta| \leq \frac{1-\upsilon}{[\eta(\rho+1)-(\upsilon+\rho)]\phi(\lambda,t,\eta)}, \quad \eta \geq 2.$$

We may set

$$\zeta_\eta = \frac{[\eta(\rho+1)-(\upsilon+\rho)]\phi(\lambda,t,\eta)}{1-\upsilon} |a_\eta|, \quad \eta \geq 2$$

and $\zeta_1 = 1 - \sum_{\eta=2}^{\infty} \zeta_\eta$. Then □

$$\begin{aligned} u(z) &= z - \sum_{\eta=2}^{\infty} a_\eta z^\eta = z - \sum_{\eta=2}^{\infty} \zeta_\eta \frac{1-\upsilon}{[\eta(\rho+1)-(\upsilon+\rho)]\phi(\lambda,t,\eta)} z^\eta \\ &= z - \sum_{\eta=2}^{\infty} \zeta_\eta [z - u_\eta(z)] = z - \sum_{\eta=2}^{\infty} \zeta_\eta z + \sum_{\eta=2}^{\infty} \zeta_\eta u_\eta(z) \\ &= \zeta_1 u_1(z) + \sum_{\eta=2}^{\infty} \zeta_\eta u_\eta(z) = \sum_{\eta=1}^{\infty} \zeta_\eta u_\eta(z). \end{aligned}$$

Corollary 4.1. *The extreme points of $TS(\upsilon,\rho,\lambda,t)$ are the functions $u_1(z) = z$ and*

$$u_\eta(z) = z - \frac{1-\upsilon}{[\eta(\rho+1)-(\upsilon+\rho)]\phi(\lambda,t,\eta)} z^\eta, \ \eta \geq 2.$$

5 Radii of Close-to-Convexity, Starlikeness and Convexity

A function $u \in TS(\upsilon,\rho,\lambda,t)$ is said to be close-to-convex of order δ if it satisfies

$$\Re\{u'(z)\} > \delta, \ \ (0 \leq \delta < 1; \ z \in E).$$

Also A function $u \in TS(\upsilon,\rho,\lambda,t)$ is said to be starlike of order δ if it satisfies

$$\Re\left\{ \frac{zu'(z)}{u(z)} \right\} > \delta, \ \ (0 \leq \delta < 1; \ z \in E).$$

Further a function $u \in TS(\upsilon,\rho,\lambda,t)$ is said to be convex of order δ if and only if $zu'(z)$ is starlike of order δ that is if

$$\Re\left\{ 1 + \frac{zu'(z)}{u(z)} \right\} > \delta, \ \ (0 \leq \delta < 1; \ z \in E).$$

Theorem 5.1. *Let $u \in TS(\upsilon,\rho,\lambda,t)$. Then u is close-to-convex of order δ in $|z| < R_1$, where*

$$R_1 = \inf_{k\geq 2}\left[\frac{(1-\delta)[\eta-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)}{\eta(1-\upsilon)}\right]^{\frac{1}{\eta-1}}.$$

The result is sharp with the extremal function u is given by (16).

Proof. It is sufficient to show that $|u'(z)-1| \leq 1-\delta$, for $|z| < R_1$. We have

$$|u'(z)-1| = \left|-\sum_{\eta=2}^{\infty}\eta a_\eta z^{\eta-1}\right| \leq \sum_{\eta=2}^{\infty}\eta a_\eta |z|^{\eta-1}.$$

Thus $|u'(z)-1| \leq 1-\delta$ if

$$\sum_{\eta=2}^{\infty}\frac{\eta}{1-\delta}|a_\eta||z|^{\eta-1} \leq 1. \tag{21}$$

But Theorem 2.1 confirms that

$$\sum_{\eta=2}^{\infty}\frac{[\eta(\rho+1)-(\upsilon+\rho)]\phi(\lambda,t,\eta)}{1-\upsilon}|a_\eta| \leq 1. \tag{22}$$

Hence (21) will be true if

$$\frac{\eta|z|^{\eta-1}}{1-\delta} \leq \frac{[\eta(\rho+1)-(\upsilon+\rho)]\phi(\lambda,t,\eta)}{1-\upsilon}.$$

We obtain

$$|z| \leq \left[\frac{(1-\delta)[\eta-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)}{\eta(1-\upsilon)}\right]^{\frac{1}{\eta-1}}, \eta \geq 2$$

as required. □

Theorem 5.2. *Let $u \in TS(\upsilon,\rho,\lambda,t)$. Then u is starlike of order δ in $|z| < R_2$, where*

$$R_2 = \inf_{k\geq 2}\left[\frac{(1-\delta)[\eta-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)}{(\eta-\delta)(1-\upsilon)}\right]^{\frac{1}{\eta-1}}.$$

The result is sharp with the extremal function u is given by (16).

Proof. We must show that $\left|\frac{zu'(z)}{u(z)}-1\right| \leq 1-\delta$, for $|z| < R_2$.
We have

$$\begin{aligned}\left|\frac{zu'(z)}{u(z)}-1\right| &= \left|\frac{-\sum_{\eta=2}^{\infty}(\eta-1)a_\eta z^{\eta-1}}{1-\sum_{\eta=2}^{\infty}a_\eta z^{\eta-1}}\right| \\ &\leq \frac{\sum_{\eta=2}^{\infty}(\eta-1)|a_\eta||z|^{\eta-1}}{1-\sum_{\eta=2}^{\infty}|a_\eta||z|^{\eta-1}} \\ &\leq 1-\delta. \end{aligned}\tag{23}$$

Hence (23) holds true if

$$\sum_{\eta=2}^{\infty}(\eta-1)|a_\eta||z|^{\eta-1} \leq (1-\delta)\left(1-\sum_{\eta=2}^{\infty}|a_\eta||z|^{\eta-1}\right)$$

or equivalently,

$$\sum_{\eta=2}^{\infty}\frac{\eta-\delta}{1-\delta}|a_\eta||z|^{\eta-1} \leq 1. \tag{24}$$

Hence, by using (22) and (24) will be true if

$$\frac{\eta-\delta}{1-\delta}|z|^{\eta-1} \leq \frac{[\eta(\rho+1)-(\upsilon+\rho)]\phi(\lambda,t,\eta)}{1-\upsilon}$$

$$\Rightarrow |z| \leq \left[\frac{(1-\delta)[\eta-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)}{(\eta-\delta)(1-\upsilon)}\right]^{\frac{1}{\eta-1}}, \eta \geq 2$$

which completes the proof. □

By using the same approach in the proof of Theorem 5.2, we can show that $\left|\frac{zu''(z)}{u'(z)}-1\right| \leq 1-\delta$, for $|z|<R_3$, with the aid of Theorem 2.1.
Thus we have the assertion of the following Theorem 5.3.

Theorem 5.3. *Let $u \in TS(\upsilon,\rho,\lambda,t)$. Then u is convex of order δ in $|z|<R_3$, where*

$$R_3 = \inf_{k\geq 2}\left[\frac{(1-\delta)[\eta-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)}{\eta(\eta-\delta)(1-\upsilon)}\right]^{\frac{1}{\eta-1}}.$$

The result is sharp with the extremal function u is given by (16).

6 Inclusion Theorem Involving Modified Hadamard Products

For functions

$$u_j(z) = z - \sum_{\eta=2}^{\infty}|a_{\eta,j}|z^{\eta},\ j=1,2 \tag{25}$$

in the class A, we define the modified Hadamard product $(u_1 * u_2)(z)$ of $u_1(z)$ and $u_2(z)$ given by

$$(u_1 * u_2)(z) = z - \sum_{\eta=2}^{\infty}|a_{\eta,1}||a_{\eta,2}|z^{\eta}.$$

We can prove the following.

Theorem 6.1. *Let the function $u_j,\ j=1,2$, given by (25) be in the class $TS(\upsilon,\rho,\lambda,t)$ respectively. Then $(u_1 * u_2)(z) \in TS(\upsilon,\rho,\lambda,t,\xi)$, where*

$$\xi = 1 - \frac{(1-\upsilon)^2}{(\eta+1)(2-\upsilon)(2-\upsilon+\rho)(1+\lambda)-(1-\upsilon)^2}.$$

Proof. Employing the approach used earlier by Schild and Silverman [19], we need to find the biggest ξ such that

$$\sum_{\eta=2}^{\infty}\frac{[\eta-\xi+\rho(\eta-1)]\phi(\lambda,t,\eta)}{1-\xi}|a_{\eta,1}||a_{\eta,2}| \leq 1.$$

Since $u_j \in TS(\upsilon,\rho,\lambda,t),\ j=1,2$, then we have

$$\sum_{\eta=2}^{\infty}\frac{[\eta-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)}{1-\upsilon}|a_{\eta,1}| \leq 1$$

$$\text{and } \sum_{\eta=2}^{\infty}\frac{[\eta-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)}{1-\upsilon}|a_{\eta,2}| \leq 1,$$

by the Cauchy-Schwartz inequality, we have

$$\sum_{\eta=2}^{\infty}\frac{[\eta-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)}{1-\upsilon}\sqrt{|a_{\eta,1}||a_{\eta,2}|} \leq 1.$$

Thus it is sufficient to show that

$$\frac{[\eta-\xi+\rho(\eta-1)]\phi(\lambda,t,\eta)}{1-\xi}|a_{\eta,1}||a_{\eta,2}| \leq \frac{[\eta-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)}{1-\upsilon}\sqrt{|a_{\eta,1}||a_{\eta,2}|},\ \ \eta \geq 2,$$

that is

$$\sqrt{|a_{\eta,1}||a_{\eta,2}|} \leq \frac{(1-\xi)[\eta-\upsilon+\rho(\eta-1)]}{1-\upsilon)[\eta-\xi+\rho(\eta-1)]}.$$

Note that

$$\sqrt{|a_{\eta,1}||a_{\eta,2}|} \leq \frac{(1-\upsilon)}{[\eta-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)}.$$

Consequently, we need only to prove that

$$\frac{(1-\upsilon)}{[\eta-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)} \leq \frac{(1-\xi)[\eta-\upsilon+\rho(\eta-1)]}{1-\upsilon)[\eta-\xi+\rho(\eta-1)]},\ \eta \geq 2,$$

or equivalently

$$\xi \leq 1-\frac{(\eta-1)(1+\rho)(1-\upsilon)^2}{[\eta-\upsilon+\rho(\eta-1)]^2\phi(\lambda,t,\eta)-(1-\upsilon)^2},\ \eta \geq 2.$$

Since

$$A(k) = 1-\frac{(\eta-1)(1+\rho)(1-\upsilon)^2}{[\eta-\upsilon+\rho(\eta-1)]^2\phi(\lambda,t,\eta)-(1-\upsilon)^2},\ \eta \geq 2$$

is an increasing function of $\eta, \eta \geq 2$, letting $\eta = 2$ in last equation, we obtain

$$\xi \leq A(2) = 1-\frac{(1+\rho)(1-\upsilon)^2}{[2-\upsilon+\rho]^2\phi(\lambda,t,\eta)-(1-\upsilon)^2}.$$

Finally, by taking the function given by (18), we can see that the result is sharp. □

7 Convolution and Integral Operators

Let $u(z)$ be defined by (4) and suppose that $g(z) = z-\sum_{\eta=2}^{\infty}|b_\eta|z^\eta$. Then the Hadamard product (or convolution) of $u(z)$ and $g(z)$ defined here by

$$u(z)*g(z) = (u*g)(z) = z-\sum_{\eta=2}^{\infty}|a_\eta||b_\eta|z^\eta.$$

Theorem 7.1. *Let $u \in TS(\upsilon,\rho,\lambda,t)$ and $g(z) = z-\sum_{\eta=2}^{\infty}|b_\eta|z^\eta, 0 \leq |b_\eta| \leq 1$. Then $u*g \in TS(\upsilon,\rho,\lambda,t)$.*

Proof. In view of Theorem 2.1, we have

$$\begin{aligned}&\sum_{\eta=2}^{\infty}[\eta-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)|a_\eta||b_\eta|\\ \leq&\sum_{\eta=2}^{\infty}[\eta-\upsilon+\rho(\eta-1)]\phi(\lambda,t,\eta)|a_\eta|\\ \leq&(1-\upsilon).\end{aligned}$$

□

Theorem 7.2. *Let $u \in TS(\upsilon,\rho,\lambda,t)$ and α be real number such that $\alpha > -1$. Then the function $M(z) = \frac{\alpha+1}{z^\alpha}\int_0^z t^{\alpha-1}u(t)dt$ also belongs to the class $TS(\upsilon,\rho,\lambda,t)$.*

Proof. From the representation of $M(z)$, it follows that

$$M(z) = z-\sum_{\eta=2}^{\infty}|A_\eta|z^\eta,\ \text{where } A_\eta = \left(\frac{\alpha+1}{\alpha+\eta}\right)|a_\eta|.$$

Since $\alpha > -1$, than $0 \leq A_\eta \leq |a_\eta|$. Which in view of Theorem 2.1, $M \in TS(\upsilon,\rho,\lambda,t)$. □

8 Conclusion

This research has introduced a new subclass of uniformly convex functions defined by Gegenbauer polynomials with negative coefficients and studied some basic properties of geometric function theory. Accordingly, some results related to coefficient estimates, growth and distortion properties, radii of starlike and convexity and convolution properties have also been considered, inviting future research for this field of study.

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Competing Interests

The authors have no competing interests.

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References

[1] Al - Oboudi, F., *On univalent functions defined by a generalized Sälägean operator,* Int. J. Math. Math. Sci., 27, 1429 - 1436, 2004.

[2] Al- Shaqsi, K. and Darus, M., *On certain subclass of analytic univalent functions with negative coefficients,* Applied Math. Sci., 1(23), 1121 - 1128 , 2007.

[3] Aouf, M. K. and Mostafa, A. O., *Some properties of a subclass of uniformly convex functions with negative coefficients,* Demon. Math., 41 (2), 353-370, 2008.

[4] Aqlan, E., Jahangiri, J. M. and Kulkarni, S. R., *New classes of k-uniformly convex and starlike functions,* Tamkang J. Math., 35(3), 261 - 266 , 2004.

[5] Deniz, E. and Orhan, H., *Some properties of certain subclasses of analytic functions with negative coefficients by using generalized Ruscheweyh derivative operator,* Czechoslovak Math. J., 60 (3), 699 - 713 , 2010.

[6] Duren, P.L., *Univalent Functions,* A series of Comprehensive Studies in Mathematics, vol.259, Springer, New York, , 1983 .

[7] El-Sshwah,R. M., Aouf, M. K. and Hassan, A. A. M. and Hassan, A. H., *Subclass of multivalent beta-uniformly functions with varying arguments,* J, of Math., 2013, Article ID 921543, 6 pages, 2013.

[8] Goodman, A. W., *On uniformly convex functions,* Ann. Pol. Math., 56, 87 - 92, 1991.

[9] Goodman, A. W., *On Uniformly starlike functions,* J. of Math. Anal. and Appl., 155, 364 - 370 , 1991.

[10] Kanas, S. and Wisniowska, A., *Conic regions and k-uniform convexity,* Comput. Appl. Math., 105 , 327 - 336, 1999.

[11] Kanas, S. and Wisniowska, A., *Conic domains and starlike functions,* Rev. Roum. Math. Pures Appl., 45 , 647 - 657, 2000.

[12] Magesh, N., *Certain subclasses of uniformly convex functions of order α and type β with varying arguments,* J. of the Egy. Math. Soc., 21(3), 184-189, 2013.

[13] Murugusundarmoorthy, G. and Magesh, N., *Certain subclasses of starlike functions of complex order involving generalized hypergeometric functions,* Int. J. Math. Sci., 45 , 12 pages, 2010.

[14] Pommerenke, C., *Univalent Functions,* Vandenhoeck and Ruprecht, Gottingen, , 1975.

[15] Ronning, F., *Uniformly convex functions and a corresponding class of starlike functions,* Proc. Amer. Math. Soc., 118 , 189 - 196 , 1993.

[16] Ronning, F., *Integral representions of bounded starlike functions,* Ann. Pol. Math., 60(3), 298 - 297 , 1995.

[17] Salagean, G., *Subclasses of univalent functions,* Lect. Notes in Math. (Springer Verlag), 1013 , 362-372 , 1983.

[18] Santosh, M. P., Rajkumar, N. I., Thirupathi Reddy, P. and Venkateswarlu, B. , *A new subclass of analytic functions defined by linear operator,* Adv. Math. Sci, Journal, 9(1), 205 - 217, 2020.

[19] Schild, A., Silverman, H., *Convolutions of univalent functions with negative coefficients,* Ann. Univ. Mariae Curie-Sklodowska Sect. A., 29, 99 - 107, 1975.

[20] Schober, G., *Univalent Functions,* Selected topics Lecture Notes in Math., vol. 478, Springer, New York, 1975 .

[21] Silverman, H. , *Univalent functions with negative coefficients,* Proc. Amer. Math. Soc., 51 , 109 - 116, 1975.

[22] Sobczak-Knec, M. and Zaprawa, P., *Covering domains for classes of functions with real coefficients,* Complex Var. Elliptic Equ., 52(6), 519-535, 2007.

[23] Srivastava, H. M., Shanmugam, T.N., Ramachandran, C. and Sivasubramanian, S., *A new subclass of k-uniformly convex functions with negative coefficients,* J. Inequal. Pure Appl. Math., 8(2), 1-14, 2007.

[24] Szynal, J., *An extension of typically real functions,* Ann. Univ. Mariae Curie-Sklodowska, sect. A., 48, 193–201, 1994.

[25] Venkateswarlu, B., Thirupathi Reddy, P., Sridevi, S. and Sujatha, *A certain subclass of analytic functions with negative coefficients defined by Gegenbauer polynomials,* Tatra Mountains Math. Publ., (Accepted).