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RESEARCH ARTICLE

On the Resolvent of a Non-Self-Adjoint Differential Operator in Hilbert Spaces

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Abstract

The importance of studying non-self-adjoint differential operators is becoming more and more obvious to scientists. The non-self-adjoint operators appear in many branches of science. Today, these operators have many applications in kinetic theory and quantum mechanics to linearization of equations of mathematical physics. The spectrum of these operators is unstable and their resolvent is very unpredictable. In these operators, there is no general spectral theory and this causes problems in the study of these operators. In this paper, we consider a non-self-adjoint elliptic differential operator and study its resolvent.

Keywords: resolvent; distribution of eigenvalues; non-self-adjoint operators; elliptic operators

Mathematics Subject Classification: Primary 47F05 (35JXX, 35PXX)

1 Introduction

The spectrum of self-adjoint operators is real, and a lot of valuable work has been done on this model of operators because of the general spectrum theory. Of course, these operators have many and varied applications in other sciences. But the fact is that there are many operators that are not self-adjoint. On the other hand, the study of these operators has proven to be important in the physical sciences, including quantum and thermodynamics.This article is based on the work of K.Kh. Boimatov [2, 3, 4, 5, 6]. We have generalized the operator used as well as the method used by him. A. Samaripour's research work has also been used in writing this article [10, 11, 16]. To get a feeling for the history of the subject under study, refer to papers [10, 11, 14, 16]. Indeed this paper was written in continuing on earlier our papers, the paper is sufficiently more general than earlier our papers, which here, we obtain the resolvent estimate of the operator P, that satisfying the special and general conditions, This operator is a second-order elliptic differential operator and is more general than the Sturm-Liouville operators and Schrödinger operators. We consider the weighted Sobolev space $\mathcal{H}_\ell = W^2_{2,\alpha}(0,1) \times W^2_{2,\alpha}(0,1) \times ... \times W^2_{2,\alpha}(0,1)$ (ℓ-times) as the space of vector functions $u(t) = (u_1(t), \ldots, u_\ell(t))$ defined on $(0,1)$ with the finite norm $|u|_+ = \left(\int\limits_0^1 e^{2\alpha t}\left|\frac{du(t)}{dt}\right|^2_{\mathbf{C}^\ell} + \int\limits_0^1 |u(t)|^2_{\mathbf{C}^\ell}\,dt\right)^{\frac{1}{2}}$. Here the notions $|\frac{du(t)}{dt}|^2_{\mathbf{C}^\ell}$, $|u(t)|^2_{\mathbf{C}^\ell}$ stand for the norm in space $\mathbf{C}^\ell$. The above definition of norm has been previously used. (See [2], [3],[5]). By $\overset{\circ}{\mathcal{H}}_\ell$ we denote the closure of $C_0^\infty(0,1)^\ell$ with respect to the above norm (i.e., $\overset{\circ}{\mathcal{H}}_\ell$ is closure of $C_0^\infty(0,1)^\ell$ in the $\mathcal{H}_\ell$). $C_0^\infty(0,1)$ denotes the space of infinitely differentiable functions with compact support in $(0,1)$. If $\ell = 1$, then, $H = H_1$, $\mathcal{H} = \mathcal{H}_1$, and $\overset{\circ}{\mathcal{H}} = \overset{\circ}{\mathcal{H}}_1$. We now consider a non-selfadjoint elliptic differential operator of type $(Pu)(t) = -\frac{d}{dt}\left(e^{2\alpha t} q(t)\frac{du(t)}{dt}\right)$, acting in space $H_\ell = L^2(0,1)^\ell$ with Dirichelet-type boundary conditions. Here $0 \le \alpha < 1,\ t \in [0,1],\ q(t) \in C^2\left([0,1], End\ \mathbf{C}^\ell\right)$. Assume that for each $t \in [0,1]$ the matrix function $q(t)$ has ℓ-simple non-zero eigenvalues $\mu_1(t), \ldots, \mu_\ell(t)$ in the complex plane. Let $\mu_j(t) \in C^2([0,1], \mathbf{C})$ $j = 1, \ldots, \ell$. Moreover, let $\Phi = \{z \in \mathbf{C} : |arg\ z| \le \varphi\},\ \varphi \in (0, \pi)$ be some closed sector with vertex at zero. We now consider different locations for the eigenvalues $\mu_j(t)$ in view of $\Phi \subset \mathbf{C}$. For a closed extension of the operator P according to space $H_\ell = W^2_{2,\alpha}(0,1)^\ell$ above, we need to extend its domain to the closed domain $D(P) \subset \{u \in \overset{\circ}{\mathcal{H}}_\ell \cap W^2_{2,\mathrm{loc}}(0,1)^\ell :\ e^{\alpha t}u'(t) \in$

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$H_\ell,\ \frac{d}{dt}\left(e^{2\alpha t}q(t)\frac{du(t)}{dt}\right)\in H_\ell\}$ (see [7]). Here $W^2_{2,\,loc}(0,1)^\ell = W^2_{2,\,loc}(0,1)\times\cdots\times W^2_{2,\,loc}(0,1)$ $(\ell - times)$ where $W^2_{2,\,loc}(0,1)$ is the space of the functions $u(t)$ $(0<t<1)$ in the form

$$\sum_{i=0}^{2}\int_{\varepsilon}^{1-\varepsilon}|u^{(i)}(t)|^2dt<\infty. \qquad \forall\varepsilon\in(0,\frac{1}{2}).$$

In this article, we investigate the spectral properties, and the asymptotic distribution of eigenvalues of the non- self adjoint differential operator P, which is defined in space $H_\ell = L^2(0,1)^\ell$ for two boundary conditions. Here, and in the sequel, the value of the function $\arg z\in(-\pi,\pi]$ and $\|T\|$ denotes the norm of the bounded arbitrary operator T acting in H or H_ℓ. Assume that for each $t\in[0,1]$ the matrix function $q(t)$ has ℓ-simple non-zero eigenvalues $\mu_j(t)\in C^2[0,1],\ j=1,\ldots,\ell$, arranged in the complex plane in the following way:

$$\mu_1(t),\ldots,\mu_{\nu(t)}\in R_+,\ \mu_{\nu+1}(t),\ldots,\mu_\ell(t)\in\mathbf{C}\backslash\Phi,$$

then, we will estimate the resolvent of differential operator P, acting in space $H_\ell = L^2(0,1)^\ell$.

2 On the resolvent estimate of the differential operator A in $H = L^2(0,1)$

Theorem 2.1. *Let $\Phi\subset\mathbf{C}$ be some closed sector with vertex at 0, and set $P=A,\ q(t)=\mu(t)$ in the definition of the operator P, as in Section 1. Then, we obtain $(Av)(t) = -\frac{d}{dt}\left(e^{2\alpha t}\mu(t)\frac{dv(t)}{dt}\right)$, acting in $H = L_2(0,1)$.*
Assume that $\mu(t)\in C\backslash\Phi$,$t\in[0,1]$. Then, for sufficiently large in modulus $\lambda\in\Phi$, the inverse $(A-\lambda I)^{-1}$ exists and is continuous in the space $H=L^2(0,1)$, and the following estimates hold;

$$\|(A-\lambda I)^{-1}\|\le M_\Phi|\lambda|^{-1}\ (\lambda\in\Phi,\ \ |\lambda|>C_\Phi), \tag{2.1}$$

$$\|e^{\alpha t}\ \frac{d}{dt}(A-\lambda I)^{-1}\|\le M'_\Phi|\lambda|^{-\frac{1}{2}}\ (\lambda\in\Phi,\ \ |\lambda|>C_\Phi), \tag{2.2}$$

where the numbers M_Φ, M'_Φ and $C_\Phi>0$ are sufficiently large numbers depending on Φ where $\Phi=\{z\in\mathbf{C}:\ |arg\ z|\le\varphi\}$, $\varphi\in(0,\pi)$

Proof. Here, to establish Theorem 2.1, we will first prove the assertion of Theorem 2.1 together with estimate (2.1). As in Section 1, for the closed extension the operator A, (for more explain see chapter 6 of [8]), we need to extend its domain to the closed set $D(A)=\{v\in\overset{\circ}{\mathcal{H}}\cap W^2_{2,\mathrm{loc}}(0,1):\ \ e^{\alpha t}u'(t)\in H,\ \left(e^{2\alpha t}\mu(t)v'(t)\right)'\in H\}$. There exists a real $\gamma\in(-\pi,\pi]$, such that for the complex number $e^{i\gamma}$ we have:

$$c'\le Re\{e^{i\gamma}\mu(t)\},\ \ c'|\lambda|\le -Re\{e^{i\gamma}\lambda\},\quad c'>0\ ,t\in[0,1],\ \lambda\in\Phi \tag{2.3}$$

let $h(t)=e^{2\alpha t}$ by integrating from both sides of $c'\le Re\{e^{i\gamma}\mu(t)\}$ for $v\in D(A)$ we will have

$$c'\int_0^1 h(t)|v'(t)|^2dt\le Re\int_0^1 e^{i\gamma}h(t)\mu|v'(t)|^2\,dt = Re\{e^{i\gamma}(Av\,,\,v)\}. \tag{2.4}$$

Here the symbol (,) denotes the inner product in H. Notice that the above equality in (2.4) obtains by the well known theorem of the m-sectorial operators, which are closed by extending its domain to the closed domain in $\mathcal{H}$. These operators are associated with the closed sectorial bilinear forms that are densely defined in $\mathcal{H}$ (for further explanation see the well known Theorem 2.1, chapter 6 of [8]). This is why, we extend the domain of the operator A to the closed domain in space $\mathcal{H}$, above.
By (2.3), we have: $c'|\lambda|\le -Re\{e^{i\gamma}\lambda\}$, $c'>0$, $\forall\lambda\in\Phi$. Multiplying the latter inequality by $\int_0^1|v(t)|^2\,dt=(v\,,\,v)=\|v\|^2>0$, then

$$c'|\lambda|\int_0^1|v(t)|^2\,dt\le -Re\{e^{i\gamma}\lambda\}(v\,,\,v).$$

By the latter inequality, and (2.4), and by considering $c'=1/M$, it follows that

$$\begin{aligned}\int_0^1 h(t)|v'(t)|^2dt+|\lambda|\int_0^1|v(t)|^2dt &\le MRe\{e^{i\gamma}(A,v)-e^{i\gamma}\lambda(v,v)\}\\ &= MRe\{e^{i\gamma}((A-\lambda I)v,v)\}\end{aligned}$$

$$\begin{aligned} &\leq M\|e^{i\gamma}\|\|v\|\|(A-\lambda I)v\| \\ &= M\|v\|\|(A-\lambda I)v\|. \quad (2.5) \end{aligned}$$

Or

$$\int_0^1 h(t)|v'(t)|^2 dt + |\lambda| \int_0^1 |v(t)|^2\, dt \leq M\|v\|\ \|(A\ -\ \lambda\ I)v\|.$$

Since, $\int_0^1 h(t)|v'(t)|^2 dt$ is positive, we will have either

$$|\lambda|\|v(t)\|^2 = |\lambda| \int_0^1 |v(t)|^2\, dt \leq M\|v\|\|(A\ \ -\lambda\ I)v\|, \tag{2.6}$$

i.e.,

$$|\lambda|\|v(t)\| \leq M\|(A\ \ -\lambda\ I)v\|.$$

The above relation ensures that the operator $(A\ -\ \lambda\ I)$ is one- to- one, which implies that $ker(A\ -\ \lambda\ I) = 0$. Therefore, the inverse operator $(A-\ \ \lambda I)^{-1}$ exists and its continuity follows from the proof of the estimate (2.1) of Theorem 2.1. To prove (2.1), we set $v = (A-\lambda I)^{-1}f,\ f \in H$ in (2.6), so that

$$|\lambda| \int_0^1 |(A-\lambda I)^{-1}f|^2\, dt \leq M\|(A-\ \ \lambda I)^{-1}f\|\|(A\ \ -\lambda I)(A-\lambda I)^{-1}f\|.$$

Since, $(A\ -\ \lambda\ I)(A-\lambda I)^{-1}f = I(f) = f$ it follows that

$$|\lambda| \int_0^1 |(A\ \ -\lambda I)^{-1}f|^2\, dt \leq M\|(A\ \ -\lambda I)^{-1}f\||f|.$$

Therefore,

$$|\lambda|\|(A\ \ -\lambda I)^{-1}(f)\|^2 \leq M\|(A\ \ -\lambda I)^{-1}(f)\||f|.$$

By canceling the positive term $\|(A-\lambda I)^{-1}(f)\|$ from both sides of the latter inequality we will find

$$|\lambda|\|(A\ \ -\lambda I)^{-1}(f)\| \leq M|f|,$$

and since $\lambda \neq 0$, we imply that $\|(A-\lambda I)^{-1}(f)\| \leq M|\lambda|^{-1}|f|$.The end result is

$$\|(A-\lambda I)^{-1}\| \leq M_\Phi|\lambda|^{-1}.$$

This estimate completes the proof of the assertion Theorem 2.1, together with the estimate (2.1).

To complete the proof of Theorem 2.1, here, we must prove estimate (2.2), therefore, we begin to prove the estimate (2.2) of Theorem 2.1. As in the first arguments to prove estimate (2.1) above, here, we drop the positive term $|\lambda| \int_0^1 |v(t)|^2\, dt$ from

$$\int_0^1 h(t)|v'(t)|^2 dt + |\lambda| \int_0^1 |v(t)|^2\, dt \leq M|v|\ \|(A\ -\ \lambda\ I)v\|.$$

It follows that

$$\int_0^1 h(t)|v'(t)|^2 dt \leq M|v|\ \|(A\ -\ \lambda\ I)v\|.$$

Set $v = (A-\lambda I)^{-1}f,\ f \in H$ in the latter inequality, and as above, by proceeding with similar calculations, we then obtain:

$$\int_0^1 h(t)|\frac{d}{dt}(A-\lambda I)^{-1}f(t)|^2 dt \leq M\|(A-\lambda I)^{-1}f\|\|(A\ -\ \lambda\ I)(A-\lambda I)^{-1}f\|.$$

Since, $(A\ -\ \lambda\ I)(A-\lambda I)^{-1}f = f$, and

$$\int_0^1 h(t)|\frac{d}{dt}(A-\lambda I)^{-1}f(t)|^2 dt \leq M\|(A-\lambda I)^{-1}f\||f|,$$

consequently by (2.1) we have $\|(A-\lambda I)^{-1}f\| \leq M|f||\lambda|^{-1}$. Then,

$$\int_0^1 h(t)|\frac{d}{dt}(A-\lambda I)^{-1}f(t)|^2 dt \leq M\|(A-\lambda I)^{-1}f\||f| \leq MM|\lambda|^{-1}|f|^2.$$

Therefore,

$$\int_0^1 h(t)|\frac{d}{dt}(A-\lambda I)^{-1}f(t)|^2 dt \leq M'_\Phi|\lambda|^{-1}|f|^2;$$

i.e., $\|h^{\frac{1}{2}}\ \frac{d}{dt}(A-\lambda I)^{-1}f\|^2 \leq M'_\Phi|\lambda|^{-1}|f|^2$. i.e.,

$$\|h^{\frac{1}{2}}\ \frac{d}{dt}(A-\lambda I)^{-1}f\| \leq M'_\Phi|\lambda|^{-\frac{1}{2}}|f|.$$

Consequently,

$$\|h^{\frac{1}{2}}\ \frac{d}{dt}(A-\lambda I)^{-1}\| \leq M'_\Phi|\lambda|^{-\frac{1}{2}}.$$

This estimate completes the proof of (2.2); Theorem 2.1 is thereby proved. □

3 On the resolvent estimate of the differential operator in H_ℓ

Theorem 3.1. *Let P, $h(t)$, $q(t)$, Φ be defined as in Section 1. Suppose that for each $t\in[0,1]$ the matrix function $q(t)$ has ℓ- distinct non-zero simple eigenvalues $\mu_j(t)\in C^2[0,1](1\leq j\leq \ell)$ located in the complex plane $\mathbf{C}$, in the form*

$$\mu_1(t),\dots,\mu_\ell(t)\in \mathbf{C}\backslash\Phi. \tag{3.1}$$

As in Theorem 2.1, but here we let the operator P acting in the space $H_\ell = L^2(0,1)^\ell$ with Dirichelet-type boundary conditions. Then, for sufficiently large in modulus $\lambda\in\Phi$, the inverse operator $(P-\lambda I)^{-1}$ exists and is continuous in the space $H_\ell = L^2(0,1)^\ell$ and the following estimate holds;

$$\|(P-\lambda I)^{-1}\| \leq M_\Phi|\lambda|^{-1} \tag{3.2}$$

where M_Φ, $C_\Phi>0$ is sufficiently large number depending on Φ and $|\lambda|>C_\Phi$.

Proof. The conditions which we consider on the eigenvalues $\mu_j(t)$ of the matrix function $q(t)$ guarantee that one can convert the matrix $q(t)$ to the diagonal form

$$q(t) = U(t)\Lambda(t)U^{-1}(t),\ \text{where } U(t),U^{-1}(t)\in C^2([0,1],End\ \mathbf{C}^\ell)$$

where $\Lambda(t) = diag\{\mu_1(t),\dots,\mu_\ell(t)\}$. Consider space $H_\ell = H\oplus\cdots\oplus H$ (ℓ-times). Let us introduce the operator

$$B(\lambda) = diag\{(P_1-\lambda I)^{-1},\dots,(P_\ell-\lambda I)^{-1}\}, \tag{3.3}$$

acting on the space H_ℓ. Due to results of Theorem 2.1, if we set $P_j = A$ for $j=1,\dots,\ell$, then, here as previously, we will have

$$(P_j v)(t) = -\frac{d}{dt}\left(h\mu_j\frac{dv(t)}{dt}\right),$$

$$D(P_j) = \{v\in\overset{\circ}{\mathcal{H}}\cap W^2_{2,\,loc}(0,1):\ h^{\frac{1}{2}}u'\in H,\ h^{\frac{1}{2}}u'\in H,\ \frac{d}{dt}\left(h\mu_j\frac{du}{dt}\right)\in H\}.$$

According to the results that obtained in Section 2, it follows the operator $B(\lambda)$, defined in (3.3), exists and is continuous for sufficiently large absolute values of $\lambda\in\Phi$. We consider the operator $\Gamma(\lambda) = UB(\lambda)U^{-1}$ where $(Uu)(t) = U(t)u(t)$, $(u\in H_\ell)$. Consequently, it follows that

$$(P-\lambda I)\Gamma(\lambda)u = -\frac{d}{dt}\left(h(t)q(t)\frac{d}{dt}(U(t)B(\lambda)U^{-1}(t)u(t))\right) = T_1+T_2+T_3$$

where

$$T_1 = -\frac{d}{dt}\left(h(t)q(t)U(t)\frac{d}{dt}B(\lambda)U^{-1}(t)u(t)\right) = -\frac{d}{dt}(h^{\frac{1}{2}}(t)U(t)\Lambda(t)\frac{d}{dt}B(\lambda)U^{-1}u(t))$$

$$= -U\frac{d}{dt}(h^{\frac{1}{2}}(t)\Lambda(t)\frac{d}{dt}B(\lambda)U^{-1}u) - U'(t)h(t)\Lambda\frac{d}{dt}B(\lambda)U^{-1}u.$$

Using the equality

$$-\frac{d}{dt}h(t)(\Lambda\frac{d}{dt}B(\lambda)V) = V+\lambda B(\lambda)V,\quad V = U^{-1}u.$$

We can then write

$$-U\frac{d}{dt}(h(t)\Lambda(t)\frac{d}{dt}B(\lambda)U^{-1}u) = -U(V+\lambda B(\lambda)V),\ V = U^{-1}u.$$

Thus, we have

$$T_1 = \lambda UB(\lambda)U^{-1}u - U'(t)h(t)\Lambda\frac{d}{dt}B(\lambda)U^{-1}u + UU^{-1}u.$$

$$T_2 = -\frac{d}{dt}\left(h^{\frac{1}{2}}qU'B(\lambda)U^{-1}u\right), \quad T_3 = -\lambda U(t)B(\lambda)U^{-1}u.$$

Using (2.1), (2.2), and from the above relations, we will have

$$(P-\lambda I)\Gamma(\lambda) = I + T_1^0 + T_2^0 \quad \text{where } T_2^0 = (h)'RU'B(\lambda)U^{-1},$$

$$\|T_1^0\| \leq M|\lambda|^{-1/2} \quad (\lambda \in \Phi,\ |\lambda| \geq C_\Phi).$$

By the Hardy-type inequality (see Chapter 6 of [8]), we have

$$\int_0^1 t^{-1+\varepsilon'_1}(1-t)^{-1+\varepsilon'_2}|y(t)|^2 dt \leq M(\varepsilon'_1,\ \varepsilon'_2)\int_0^1 |y(t)|^2\, dt$$

$$+M(\varepsilon'_1,\ \varepsilon'_2)\int_0^1 t^{1+\varepsilon'_1}(1-t)^{1+\varepsilon'_2}|y'(t)|^2\, dt, \quad \forall y \in \overset{\circ}{\mathcal{H}},\ \ \varepsilon'_1,\ \varepsilon'_2 \neq 0.$$

Since $|q(t)U'(t)|_{\mathbf{C}^\ell \to \mathbf{C}^\ell} \leq M$, and by $(1.2')$, we will estimate the operator T_2^0 as follows: Using the Hardy-type inequality above, and taking $W = B(\lambda)u$, then we have

$$\begin{aligned}
\int_0^1 |h'(t)|^2|(B(\lambda)u)(t)|^2_{\mathbf{C}^\ell}dt &= \int_0^1 |h'(t)|^2|W(t)|^2_{\mathbf{C}^\ell}dt \\
&\leq M_2\int_0^1 t^{\alpha-2+2\varepsilon'_1}(1-t)^{\beta-2+2\varepsilon'_2}|W(t)|^2_{\mathbf{C}^\ell}dt \\
&\leq M_3\int_0^1 t^{\alpha}(1-t)^{\beta}t^{2\varepsilon'_1}(1-t)^{2\varepsilon'_2}\,|W'(t)|^2_{\mathbf{C}^\ell}dt \\
&+ M||(B(\lambda)u)||^2_{H_\ell}.
\end{aligned}$$

i.e.

$$\int_0^1 |h'(t)|^2|(B(\lambda)u)(t)|^2{}_{\mathbf{C}^\ell}dt \ \leq\ M_3\int_0^1 t^{\alpha}(1-t)^{\beta}\ \ t^{2\varepsilon'_1}(1-t)^{2\varepsilon'_2}\ |W'(t)|^2_{\mathbf{C}^\ell}dt\ + M||(B(\lambda)u)||^2_{H_\ell}.$$

Since $t^{2\varepsilon'_1}(1-t)^{2\varepsilon'_2} \leq M'$, by the estimates (2.1) and (2.2), the last inequality is equivalent to

$$\begin{aligned}
\int_0^1 |h'(t)|^2|(B(\lambda)u)(t)|^2_{\mathbf{C}^\ell}\, dt &\leq M_3M'\int_0^1 t^{\alpha}(1-t)^{\beta}|W'(t)|^2_{\mathbf{C}^\ell}\, dt \\
&+ M||(B(\lambda)u)||^2_{H_\ell} \\
&\leq M_3M'\int_0^1 h(t)|W'(t)|^2_{\mathbf{C}^\ell}\, dt \\
&+ M||(B(\lambda)u)||^2_{H_\ell}
\end{aligned}$$

As in Section 2, for finding the estimates (2.1), (2.2) we have

$$\begin{aligned}
\int_0^1 |h'(t)|^2|(B(\lambda)u)(t)|^2_{\mathbf{C}^\ell}dt &\leq M'_3\int_0^1 h(t)|W'(t)|^2_{\mathbf{C}^\ell}dt \\
&+ M||(B(\lambda)u)||^2_{H_\ell} \\
&\leq M'_3\int_0^1 h(t)|W'(t)|^2_{\mathbf{C}^\ell}\, dt \\
&+ M||(B(\lambda)u)||^2_{H_\ell} \\
&\leq M'_3|\lambda|^{-1/2} + M|\lambda|^{-1} \\
&\leq M_\Phi|\lambda|^{-1/2}
\end{aligned}$$

Then, $\|T_2^0\| \leq M_\Phi|\lambda|^{-1/2}$ for sufficiently large absolute-value of $\lambda \in \Phi$. Consequently,

$$(P-\lambda I)\Gamma(\lambda) = I+\mathcal{F}(\lambda), \quad \|\mathcal{F}(\lambda)\| \leq M|\lambda|^{-\frac{1}{2}} \quad (\lambda \in \Phi, \ |\lambda| > C'). \tag{3.4}$$

since $\|\mathcal{F}(\lambda)\| \leq M|\lambda|^{-\frac{1}{2}}$, it easily follows that $I+\mathcal{F}(\lambda)$ and so, by (3.4) $(A-\lambda I)\Gamma(\lambda)$ is invertible. We then have

$$((P-\lambda I)\Gamma(\lambda))^{-1} = (I+\mathcal{F}(\lambda))^{-1}.$$

By adding $+I$ and $-I$ to the right side of the relation above we will have

$$(I+\mathcal{F}(\lambda))^{-1} = (I \ + \ F(\lambda))^{-1} \ - \ I \ +I$$

If $\mathcal{Y}(\lambda) = (I \ + \ F(\lambda))^{-1} \ - \ I$, since $\|\mathcal{F}(\lambda)\| \leq M|\lambda|^{-\frac{1}{2}}$, this implies

$$\|\mathcal{Y}(\lambda)\| \leq M2\Phi|\lambda|^{-1} \quad (\lambda \in \Phi, \ |\lambda| > C). \tag{3.5}$$

Consequently,

$$(P-\lambda I)^{-1} = \Gamma(\lambda)(I+\mathcal{Y}(\lambda)), \tag{3.6}$$

since

$$\Gamma(\lambda) = UB(\lambda)U^{-1},\ B(\lambda) = diag\{(P_1-\lambda I)^{-1},\ldots,(P_\ell-\lambda I)^{-1}\}. \tag{3.7}$$

Putting $A_j = P_j,\ j = 1,\ldots,\ell$,then $\|(P_j-\lambda I)^{-1}\| \leq M_\Phi|\lambda|^{-1}$, for $j = 1,\ldots,\ell$. From it follows that $\|\Gamma(\lambda)\| \leq M1_\Phi|\lambda|^{-1}$. Then

$$\begin{aligned} \|(P-\lambda I)^{-1}\| &\leq \|\Gamma(\lambda)\|\|(I+\mathcal{Y}(\lambda))\| \\ &\leq M1_\Phi|\lambda|^{-1}(1+M2_\Phi|\lambda|^{-1}) \\ &\leq M_\Phi|\lambda|^{-1}. \end{aligned}$$

i.e.,

$$\|(P-\lambda I)^{-1}\| \leq M_\Phi|\lambda|^{-1}$$

where M_Φ is a sufficiently large number depending on Φ. This completes the proof of Theorem 3.1. □

Competing Interests

The authors have no competing interests.

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